

# THREEFOLDS WITH BIG AND NEF ANTICANONICAL BUNDLES II

PRISKA JAHNKE, THOMAS PETERNELL, AND IVO RADLOFF

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## 1. INTRODUCTION

This is the second part of our classification of smooth complex projective threefolds  $X$  whose anticanonical bundles  $-K_X$  are big and nef, but not ample. In other words, we classify *almost Fano threefolds*. In the first part [JPR05] we classified those  $X$  with Picard number  $\rho(X) = 2$ , whose anticanonical morphism contracts a divisor. To be more specific, recall that some multiple  $-mK_X$  is generated by global sections (usually  $m = 1$ ) inducing a morphism with connected fibers

$$\psi : X \rightarrow X'$$

which we call the *anticanonical morphism* of  $X$ . Since  $-K_X$  is big but not ample,  $\psi$  is birational, but not an isomorphism.

In this paper we are concerned with the case  $\rho(X) = 2$  and  $\psi$  *small*, i.e.,  $\psi$  contracts finitely many curves and nothing else. The singular variety  $X'$  is Fano with terminal Gorenstein singularities, but it is not  $\mathbb{Q}$ -factorial. By [Na97],  $X'$  admits a smoothing  $(X_t)$ , and  $\rho(X_t) = 1$  by [JR06a]. In this situation  $X$  can be flopped, i.e., there is another almost Fano threefold  $X^+$  with small anticanonical morphism  $\psi^+ : X^+ \rightarrow X'$  which henceforth is isomorphic to  $X$  in codimension 1. This is an important tool to study the original threefold  $X$ . Another main ingredient is the unique Mori contraction  $\phi : X \rightarrow Y$ , which is somehow perpendicular to  $\psi$ . Since

$\rho(X^+) = 2$ , too, also  $X^+$  carries a unique Mori contraction  $\psi^+ : X^+ \rightarrow Y^+$ , and we study the interplay of the two contractions.

Our classification results are resumed in the lists in the appendix. We classify according to the types of  $\phi$  and  $\phi^+$ : del Pezzo fibrations over  $\mathbb{P}_1$  (including projective and quadric bundles), conic bundles over  $\mathbb{P}_2$ , and birational contractions. However, due to the complexity of the problem and the length of the paper, we shall not consider the case that both contractions are birational, and hope to come back to that case later.

In many cases we explicitly write down examples, but in some circumstances this is very delicate and must be left open. The reason for that is twofold. First, it is difficult to construct explicitly del Pezzo fibrations with relative Picard number 2 and  $K_F^2 = 5, 6$  (with  $F$  the general fiber) and second, in the case of conic bundles  $X \subset \mathbb{P}(E)$ , it is possible to write down the potential rank 3 bundles  $E$  over  $\mathbb{P}_2$ , but in order to construct  $X$ , it is necessary to work out “smooth” sections in a certain twist of  $S^2(E)$  which is difficult, too.

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## 2. PRELIMINARIES

**2.1. Notation.** As in Part I we consider a smooth projective threefold  $X$  with  $-K_X$  big and nef. We always assume that  $X$  is *not Fano* and say that  $X$  is *almost Fano*. Then  $-mK_X$  will be spanned for suitable large  $m$ . Throughout this paper  $\psi : X \rightarrow X'$  will denote the morphism (with connected fibers) associated with  $|-mK_X|$  **and  $\psi$  is assumed to be small**, therefore contracts only finitely many smooth rational curves and nothing else. Notice that  $X'$  is Gorenstein Fano threefold with only terminal singularities and  $\rho(X') = 1$ , but  $X'$  is not  $\mathbb{Q}$ -factorial.

By [Ko89], there exists the following flop-diagram

$$(2.1.1) \quad \begin{array}{ccccc} X & \overset{\chi}{\dashrightarrow} & X^+ \\ \phi \downarrow & \searrow \psi & \swarrow \psi^+ & \downarrow \phi^+ \\ & X' & & \tilde{Y} \end{array}$$

where the rational map  $\chi$  is an isomorphism outside the exceptional locus of  $\psi$  and  $X^+$  is again a smooth almost Fano threefold with anticanonical map  $\psi^+$  and extremal contraction  $\phi^+$ . Note that  $\phi$  and  $\phi^+$  are not necessarily of the same type. Our assumption  $\rho(X) = 2$  implies that  $\chi$  does not depend on the choice of some  $\psi$ -negative divisor in  $X$ . To be more precise, we have the following

**2.2. Proposition.** *Let  $D$  be any divisor which is not  $\psi$ -nef, i.e.  $-D$  is  $\psi$ -ample. Then the  $D$ -flop of  $\psi$  exists, i.e. a small birational map  $\psi^+ : X^+ \rightarrow X'$  such that the strict transform  $\tilde{D} \subset X^+$  is  $\psi^+$ -ample. Moreover  $X^+$  is smooth with  $-K_{X^+}$  big and nef and*

$$\begin{aligned} \rho(X^+) &= 2, \\ (-K_X)^3 &= (-K_{X^+})^3, \\ h^0(\mathcal{O}_X(D)) &= h^0(\mathcal{O}_{X^+}(\tilde{D})), \end{aligned}$$

the same being true for the strict transform of any divisor on  $X$ . Finally  $\psi^+$  does not depend on  $D$ .

*Proof.* Let  $l$  be a curve contracted by  $\psi$ . Then  $D \cdot l < 0$ , hence the  $D$ -flop exists by [Ko89]. Also the smoothness and the statement on the Picard number follows from [Ko89]. Since  $\psi^+$  is small and since  $X'$  has only terminal singularities, we have  $K_{X^+} = (\psi^+)^*(K_{X'})$ , hence  $-K_{X^+}$  is big and nef and also  $(-K_X)^3 = (-K_{X^+})^3$ . The  $H^0$ -statement is clear, too. Finally  $\psi^+$  does not depend on  $D$ , since  $\rho(X) = 2$  so that two divisors  $D$  and  $D'$  which are negative on the curves  $l_\psi$  coincide up to multiples in a neighborhood of the exceptional locus of  $\psi$ .  $\square$

**2.3. Notation.** The  $D$ -flop as in (2.1.1) will always denoted  $\psi^+ : X^+ \rightarrow X'$ ; if  $E$  is a divisor on  $X$ , then the strict transform of  $E$  will be denoted by  $\tilde{E}$ . On the level of sheaves, let  $L$  be the pull back to  $X$  of the ample generator on  $Y$ ; then we set

$$\tilde{L} = (\psi^+)^*(\psi_*(L)^{**}).$$

Notice that  $-\tilde{K}_X = -K_{X^+}$ . The induced birational map  $X \dashrightarrow X^+$  is called  $\chi$ . Since  $\rho(X^+) = 2$  and since  $-K_{X^+}$  is big and nef but not ample,  $X^+$  carries a unique contraction which is called  $\phi^+ : X^+ \rightarrow Y^+$ . Then as above we consider the pull back  $L^+$  of an ample generator on  $Y^+$  and define  $\tilde{L}^+$  a line bundle on  $X$ .

A *smoothing* of a singular Fano threefold  $X'$  is a flat family

$$\mathcal{X} \longrightarrow \Delta$$

over the unit disc, such that  $\mathcal{X}_0 \simeq X'$  and  $\mathcal{X}_t$  is a smooth Fano threefold for  $t \neq 0$ . Namikawa has shown in [Na97] that a smoothing always exists if  $X'$  has only terminal Gorenstein singularities, not necessarily  $\mathbb{Q}$ -factorial: In this case the Picard groups of  $X'$  and the general  $\mathcal{X}_t$  are isomorphic (over  $\mathbb{Z}$ ) by [JR06a].

**2.4. Proposition** [[Na97], [JR06a]]. *Let  $X'$  be a Gorenstein Fano threefold with only terminal singularities (not necessarily  $\mathbb{Q}$ -factorial). Then  $X'$  has a smoothing  $\mathcal{X} \rightarrow \Delta$  and  $\text{Pic}(X') \simeq \text{Pic}(\mathcal{X}_t)$ . In particular,  $X'$  and  $\mathcal{X}_t$  have the same Picard number and the same index.*

**2.5. Proposition.** *The anticanonical bundle  $-K_{X'}$  and therefore  $-K_X$  are generated by global sections unless  $X'$  is a deformation of the Fano threefold  $V_2$  and arises as complete intersection of a quadric cone and a general sextic in the weighted projective space  $\mathbb{P}(1^4, 2, 3)$ . In this case, there exists a small resolution  $X \rightarrow X'$  with  $\rho(X) = 2$ ,  $(-K_X)^3 = 2$  and  $X \simeq X^+$  both admit a del Pezzo fibration with  $K_F^2 = 1$ .*

*Proof.* Assume  $X'$  is a Gorenstein, not  $\mathbb{Q}$ -factorial Fano threefold with terminal singularities, such that  $|-K_{X'}|$  is not base point free. By [JR06b],  $X' \subset \mathbb{P}(1^4, 2, 3)$  is a complete intersection of a quadric cone  $Q$  in the first four variables and a general sextic. This means  $X'$  is a double cover of the cone over the quadric  $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1 \hookrightarrow \mathbb{P}_8$  embedded by the system  $|(2, 2)|$ , i.e. we have

$$\begin{array}{ccccc} V & \xrightarrow{2:1} & \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2, 2)) & \xrightarrow{\pi} & Q \simeq \mathbb{P}_1 \times \mathbb{P}_1 \\ \downarrow q & & \downarrow p & & \\ X' & \xrightarrow{2:1} & \hat{Q} & & \end{array}$$

Here  $V$  is a smooth almost Fano threefold with  $\rho(V) = 3$ , the anticanonical divisor  $-K_V$  being the pull back of  $\pi^*\mathcal{O}(1, 1)$ . The double cover  $V \rightarrow \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2, 2))$  is ramified along the minimal section

$$Q_0 \simeq \mathbb{P}_1 \times \mathbb{P}_1$$

of the projective bundle and a general cubic, disjoint from  $Q_0$ . The vertical maps  $p$  and  $q$  contract  $Q_0$  and its reduced inverse image  $E$  in  $V$  to a point. We have

$$K_V = q^*K_{X'} + E$$

and  $E|_E = \mathcal{O}(-1, -1)$  by adjunction formula.

Let  $\zeta$  be the tautological line bundle on  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2, 2))$  and  $F_1, F_2 \simeq \Sigma_2$  the pull-back of the two rulings of  $Q$ . The contraction  $p$  of  $Q_0$  to a point factors over the blowdown of  $Q_0$  to  $\mathbb{P}_1$  along the two rulings, defined by  $|\zeta + F_i|$ :

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2, 2)) & \xrightarrow{\pi} & Q \simeq \mathbb{P}_1 \times \mathbb{P}_1 \\ \downarrow p_i & & \downarrow \\ Z_i & \xrightarrow{\quad} & \mathbb{P}_1 \\ \downarrow & & \\ \widehat{Q} & & \end{array}$$

By construction, the maps  $p_i$  are crepant, hence  $Z_1$  and  $Z_2$  are Gorenstein almost Fano threefolds with canonical singularities along the image of  $Q_0$ . Let

$$V \xrightarrow{q_i} X_i \xrightarrow{\psi_i} X'$$

be the induced factorization of  $q : V \rightarrow X'$ , i.e.,  $X_i$  is a double cover of  $Z_i$ . Then  $q_i$  contracts  $E$  along a ruling to  $\mathbb{P}_1$ , but here  $K_V$  is negativ on the fibers, hence  $X_1$  and  $X_2$  are smooth almost Fano threefolds with  $\rho(X_i) = 2$ . The anticanonical map  $\psi_i : X_i \rightarrow X'$  is small with exceptional locus a single curve, namely  $q_i(E) = \text{Bs}|-K_{X_i}| \simeq \mathbb{P}_1$ .

On the fibers  $F_i \simeq \Sigma_2$ , the map  $p_i$  contracts the minimal section, i.e.,  $Z_i \rightarrow \mathbb{P}_1$  has general fiber the quadric cone. The induced covering gives a smooth del Pezzo surface of degree 1. The flop diagram hence is

$$\begin{array}{ccccc} X_1 & \xleftarrow{\quad} & \overset{X}{\text{---}} & \xrightarrow{\quad} & X_2 \\ & \searrow \psi_1 & & \swarrow \psi_2 & \\ & & X' & & \\ \downarrow & & & & \downarrow \\ \mathbb{P}_1 & & & & \mathbb{P}_1 \end{array}$$

with  $X = X_1 \simeq X_2 = X^+$ , but  $\chi$  not an isomorphism.  $\square$

**From now on we shall assume for the rest of the paper that  $-K_X$  is spanned.**

**2.6. Notation.** As in Part I, we let  $\mu : X' \rightarrow W$  be the finite part of the map associated with  $|-K_X|$ . We know (see e.g. Part I) that either  $\mu$  is an isomorphism

or that  $\mu$  has degree 2, in which case  $X'$  is hyperelliptic. Furthermore as usual we let  $r_X = r_{X'}$  be the index of  $X'$  and define the genus  $g$  of  $X$  or  $X'$  by

$$2g - 2 = \frac{(-K_X)^3}{2}.$$

By Bertini's classification of varieties of minimal degree ([Be07]),  $W$  is either  $\mathbb{P}_3$ , a quadric, the Veronese cone or a scroll, i.e. the image  $\overline{\mathbb{F}(d_1, d_2, d_3)}$  of a projective bundle

$$\mathbb{F}(d_1, d_2, d_3) = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(d_1) \oplus \mathcal{O}_{\mathbb{P}_1}(d_2) \oplus \mathcal{O}_{\mathbb{P}_1}(d_3)), \quad d_1 \geq d_2 \geq d_3 \geq 0$$

in  $\mathbb{P}_{d_1+d_2+d_3+2}$  under the map associated to the tautological system  $|\zeta|$ . Denote the pencil of  $\mathbb{F}(d_1, d_2, d_3)$  by  $|F|$ . We obtain

**2.7. Proposition.** *Let  $X$  be a smooth almost Fano threefold with  $\rho(X) = 2$ , such that  $\psi$  is small. If  $X' \xrightarrow{2:1} W$  is hyperelliptic, then we are in one of the following cases*

- (1)  $(-K_X)^3 = 2$ ,  $W = \mathbb{P}_3$  and  $X' \rightarrow W$  is ramified along a sextic;
- (2)  $(-K_X)^3 = 4$ ,  $W \subset \mathbb{P}_4$  is a quadric and  $X' \rightarrow W$  is ramified along a quartic;
- (3)  $(-K_X)^3 = 6$ ,  $W \subset \mathbb{P}_5$  is the singular scroll  $\mathbb{F}(2, 1, 0)$  and either  $X$  or  $X^+$  is a double cover of  $\mathbb{F}(2, 1, 0)$ , ramified along a general divisor in  $|4\zeta - 2F|$ ;
- (4)  $(-K_X)^3 = 8$ ,  $W$  is the cone in  $\mathbb{P}_6$  over the Veronese surface in  $\mathbb{P}_5$  and  $X' = X_6 \subset \mathbb{P}(1^3, 2, 3)$ , i.e.,  $X' \rightarrow W$  is ramified along a cubic.

All of these threefolds except (3) are the expected deformations of Iskovskikh's list. For (3) note, that in this case the exceptional locus of  $\psi$  consists of a single smooth rational curve, which is contained in the ramification divisor, and contracted to a point by the map  $\mathbb{F}(2, 1, 0) \rightarrow W$ . Moreover this case can be described explicitly: here  $X$  (or  $X^+$ ) admits a del Pezzo fibration with general fiber of degree 4, and a smoothing  $\mathcal{X}_t$  of  $X'$  in the sense of Namikawa is a complete intersection of a cubic and a quadric in  $\mathbb{P}_5$ . For further details and a construction of this threefold see [JR06a].

**From now on we may assume that the only hyperelliptic cases are (1), (2) and (4).**

*Proof.* If  $X'$  is hyperelliptic, then the image  $W$  of  $X'$  in  $\mathbb{P}_{g+1}$  is a variety of minimal degree of Picard number one. By Iskovskikh's classification it remains to consider

$$(2.7.1) \quad \begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ & & \downarrow \mu \\ \mathbb{F}(d_1, d_2, d_3) & \xrightarrow{\sigma} & W \end{array}$$

for some  $0 \leq d_3 \leq d_2 \leq d_1$ , i.e.  $W$  is a (singular) scroll. Then  $\rho(X') = 1$  implies  $d_3 = 0$ . If  $d_2 = 0$ , then  $W$  is a double cone over a rational normal curve of degree  $d_1$ . The double cover  $X'$  will have canonical singularities along a curve, which is impossible if  $\psi$  is small. Therefore  $d_2 > 0$ , i.e.  $W$  is a cone over a Hirzebruch surface.

Let now  $\hat{F}$  in  $X$  be the strict transform of the Weil divisor  $\sigma(F)$ . Then  $\hat{F}$  is Cartier and  $\hat{F} \cdot l_\psi \neq 0$  for any curve contracted by  $\psi$ . So after possibly replacing

$X$  by its flop  $X^+$  we may assume

$$(2.7.2) \quad \hat{F} \cdot l_\psi > 0.$$

Then  $|\hat{F}|$  is a pencil by [JR06b], Lemma 6.1 or [Ch99]. We shortly recall the argument. Assume that two general members  $F_1, F_2 \in |\hat{F}|$  are not disjoint and let  $C \subset F_1 \cap F_2$  be any irreducible curve. Then  $C \subset F_1$  is a component of the restriction of  $F_2$  to  $F_1$ , which is contained in the exceptional locus of  $\psi$ , hence contracted to points. This means  $F_2 \cdot C \leq 0$ , contradicting  $\hat{F} \cdot C > 0$  by (2.7.2).

This shows the system  $|-K_X + m\hat{F}|$  defines a factorization of  $X \rightarrow W$  over the scroll, i.e. we get a map

$$\nu: X \longrightarrow \mathbb{F}(d_1, d_2, d_3)$$

completing (2.7.1) into a commutative diagram. Considering the Stein factorization,  $\rho(X) = 2$  implies  $\nu$  is a double cover. We have  $-K_X = \nu^*\zeta$  and the exceptional locus of  $\psi$  is mapped to the exceptional curve  $C_0$  of  $\sigma$ . The ramification divisor is an element

$$D \in |4\zeta - 2(d_1 + d_2 - 2)F|.$$

We find  $C_0 \subset D$ . Moreover, for  $(d_1, d_2) \neq (1, 1), (2, 1)$ ,  $D$  will always be singular along  $C_0$ .  $\square$

Although we have a description of  $X'$  in case it is hyperelliptic, the precise structure of  $X$  itself is still not clear. The following proposition can be found in [IP99], Remark 4.1.10:

**2.8. Lemma.** *Assume  $X'$  is hyperelliptic. Denote the birational involution induced on  $X$  by  $\sigma$ . If  $W$  is  $\mathbb{Q}$ -factorial,  $\sigma$  coincides with the flop on  $X$ . In particular,  $X \simeq X^+$  as abstract varieties.*

*Proof.* Let  $D$  be some divisor on  $X$ . Denote the strict transform under  $\sigma$  by  $D^\sigma$ . Then  $D + D^\sigma$  is the pull back of some  $\sigma$ -invariant (Weil-) divisor  $B'$  on  $X'$ . Then  $B'$  comes from  $W$ . As  $W$  is  $\mathbb{Q}$ -factorial,  $mB'$  is Cartier. Then

$$(D + D^\sigma) \cdot l_\psi = \frac{1}{m} \psi^*(mB') \cdot l_\psi = 0$$

for any curve  $l_\psi$  contracted by  $\psi$ . But then  $D \cdot l_\psi = -D^\sigma \cdot l_\psi$ . This implies  $\sigma: X \dashrightarrow X$  is the flop.  $\square$

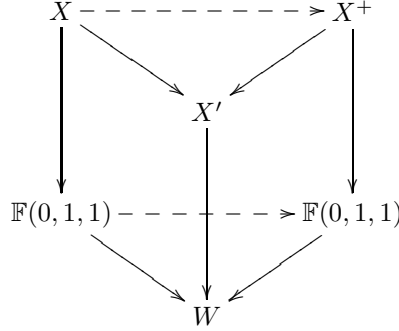
**2.9. Remark.** The same is true when  $-K_X = 2H$  with  $H$  spanned defining some double cover of some  $\mathbb{Q}$ -factorial  $W$ .

The following corollary can be found in [IP99], Remark 4.1.10:

**2.10. Corollary.** *Assume  $X'$  is hyperelliptic. Then  $X \simeq X^+$  as abstract varieties, except  $X$  is a resolution of Proposition 2.7, (3), or  $W \subset \mathbb{P}_4$  is the quadric cone. In the latter case  $(-K_X)^3 = 4$  and  $X$  as well as  $X^+$  is a double cover of  $\mathbb{F}(0, 1, 1)$ , ramified along a divisor from  $|4\zeta|$ ; they admit a del Pezzo fibration with  $K_F^2 = 2$ .*

*Proof.* Let  $X' \rightarrow W$  be the double cover defined by  $|-K_{X'}|$ . If  $W$  is  $\mathbb{Q}$ -factorial, the claim is just Lemma 2.8 above. Case (3) in Proposition 2.7 is explicitly described in [JR06a]. The only remaining case is the quadric cone. But then analogously to the proof of Proposition 2.7 either  $X$  or the flop  $X^+$  is a double cover of the small resolution  $\mathbb{F}(0, 1, 1)$  of the quadric cone.

Assume  $X \rightarrow \mathbb{F}(0, 1, 1)$  is that double cover. Since the quadric cone admits two (isomorphic) small resolutions connected by a flop, we get



meaning  $\mu$  lifts to  $X^+$  as well. The induced map  $X \rightarrow \mathbb{P}_1$  is a del Pezzo fibration, where the general fiber  $F$  is a double cover of  $\mathbb{P}_2$ , ramified along the restriction of the ramification divisor of  $X \rightarrow W_0$ , which gives a quartic. Hence  $K_F^2 = 2$ .  $\square$

For small genus we find in our situation:

**2.11. Proposition.** *Let  $X$  be a smooth almost Fano threefold with  $\rho(X) = 2$ , such that the anticanonical map  $\psi : X \rightarrow X'$  is small. Assume  $X'$  not hyperelliptic.*

- (1) *If  $g = 3$ , then  $X'_4 \subset \mathbb{P}_4$  is a quartic.*
- (2) *If  $g = 4$ , then  $X'_{2,3} \subset \mathbb{P}_5$  is a complete intersection of a quadric and a cubic.*
- (3) *If  $g = 5$ , then  $X'_{2,2,2} \subset \mathbb{P}_6$  is either a complete intersection of three quadrics or  $X \subset \mathbb{F}(1, 1, 1, 0)$  is a divisor in  $|3\zeta - F|$ . In the latter case  $X'$  is trigonal.*

*Proof.* Since the canonical curve section  $C \subset X'$  is a smooth canonical curve of genus  $g$ , (1) and (2) are easily obtained. Assume  $g = 5$ . We have two possible cases: either  $X'$  is cut out by quadrics or it is trigonal. Since  $X'$  is already a complete intersection in the first case, assume the latter one. Then by [CSP05],  $X'$  is the anticanonical model of an almost Fano threefold  $V$  with canonical singularities, where  $V$  is a divisor in  $|3\zeta - F|$  on one of  $\mathbb{F}(1, 1, 1, 0)$  or  $\mathbb{F}(2, 1, 0, 0)$ . The latter case is impossible, since here  $X'$  is singular along a curve.  $\square$

Assume now that  $-K_X$  is divisible in  $\text{Pic}(X)$ , i.e.,  $-K_X = r_X H$  for some  $H \in \text{Pic}(X)$  and  $r_X \geq 2$  the index. By assumption, then  $H$  is big and nef, hence  $|mH|$  is base point free for all  $m \gg 0$ . Since  $|mH|$  and  $|(m+1)H|$  define the same map for  $m \gg 0$ , we find

$$H = \psi^* H'$$

for some  $H' \in \text{Pic}(X')$ , and hence  $-K_{X'} = r_X H'$ . By [Shi89] then  $r_X \leq 4$ , with equality only for  $X' = \mathbb{P}_3$ , and  $r_X = 3$  implies  $X' \subset \mathbb{P}_4$  is a quadric. We obtain:

**2.12. Proposition.** *If  $r_X \geq 3$ , then  $X' \subset \mathbb{P}_4$  is the cone over a smooth quadric  $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1 \subset \mathbb{P}_3$ , and  $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus 2})$  is the small resolution of the vertex. In particular,  $X \simeq X^+$ .*

The case  $r_X = 2$  was treated in a more general situation in [JP06]; we obtain the following list for  $\rho(X) = 2$ :

**2.13. Theorem** [[JP06]]. *Assume  $\rho(X) = 2$ ,  $\psi$  is small and  $r_X = 2$ . Then  $\phi : X \rightarrow Y$  is either a quadric bundle, or a  $\mathbb{P}_1$ -bundle, or birational.*

- (1) If  $\phi$  is a quadric bundle, then  $X$  belongs to the following list.
- (i)  $X \subset \mathbb{P}_3 \times \mathbb{P}_1$  from  $|(2, 2)|$ , here  $d = 2$ ,  $X^+ \simeq X$  and  $\mathcal{X}_t \rightarrow \mathbb{P}_3$  is a double cover,
  - (ii)  $X \subset \mathbb{F}(0^3, 1)$  from  $|2\zeta + F|$ , here  $d = 3$ ,  $X^+ = \text{Bl}_p(V_{2,4})$  and  $\mathcal{X}_t \simeq V_{2,3}$  (this is case (3), (iii)),
  - (iii)  $X \subset \mathbb{F}(0^2, 1^2)$  from  $|2\zeta|$ , here  $d = 4$ ,  $X^+$  is of the same type and  $\mathcal{X}_t \simeq V_{2,4}$ ,
  - (iv)  $X \subset \mathbb{F}(0, 1^3)$  from  $|2\zeta - F|$ , here  $d = 5$ ,  $X^+ = \mathbb{P}(\mathcal{F})$  with some stable rank two bundle  $\mathcal{F} \in \mathcal{M}(-1, 2)$  (this is case (2), (i)), and  $\mathcal{X}_t \simeq V_{2,5}$ ,
- (2) If  $\phi$  is a  $\mathbb{P}_1$ -bundle, then  $X = \mathbb{P}(\mathcal{F})$  with a stable rank 2 bundle on  $\mathbb{P}_2$  with  $c_1(\mathcal{F}) = -1$  and  $2 \leq c_2(\mathcal{F}) \leq 5$ . Moreover,  $\mathcal{F}(2)$  is nef, but not ample and has only finitely many jumping lines. We have
- (i)  $c_2(\mathcal{F}) = 2$ . Then  $d = 5$ ,  $X^+$  admits a del Pezzo fibration as in (1), (iv) and  $\mathcal{X}_t \simeq V_{2,5}$ ,
  - (ii)  $c_2(\mathcal{F}) = 3$ . Then  $d = 4$ ,  $X^+ = \text{Bl}_p(V_{2,5})$  and  $\mathcal{X}_t \simeq V_{2,4}$ ,
  - (iii)  $c_2(\mathcal{F}) = 4$ . Then  $d = 3$ ,  $X^+$  is of the same type, and  $\mathcal{X}_t \simeq V_{2,3}$ ,
  - (iv)  $c_2(\mathcal{F}) = 5$ . Then  $d = 2$ ,  $X^+ \simeq X$ , and  $\mathcal{X}_t \rightarrow \mathbb{P}_3$  is a double cover.
- (3) If  $\phi$  is birational, then  $X = \text{Bl}_p(Y)$  for a general point  $p$  in a smooth del Pezzo threefold  $Y = V_{2,d+1}$ , such that
- (i)  $d = 1$ ,  $X^+ \simeq X$  and  $\mathcal{X}_t \rightarrow W$  is a double cover of the Veronese cone,
  - (ii)  $d = 2$ ,  $X^+ \simeq X$  and  $\mathcal{X}_t \rightarrow \mathbb{P}_3$  is a double cover,
  - (iii)  $d = 3$ ,  $X^+$  admits a del Pezzo fibration as in (1), (ii), and  $\mathcal{X}_t \simeq V_{2,3}$ ,
  - (iv)  $d = 4$ ,  $X^+ = \mathbb{P}(\mathcal{F})$  as in (2), (ii), and  $\mathcal{X}_t \simeq V_{2,4}$ .

From now on we will assume for the rest of the paper that  $r_X = 1$ .

### 3. DEL PEZZO FIBRATIONS

In this section we consider almost Fano threefolds admitting a del Pezzo fibration.

**3.1. Setup.** We fix for this section the following setup.  $X$  is as always a smooth projective threefold with  $-K_X$  big and nef, but not ample. Suppose that  $\phi : X \rightarrow \mathbb{P}_1$  is a del Pezzo fibration, which is the contraction of an extremal ray, i.e.  $\rho(X) = 2$ . Let  $F$  denote a general fiber of  $\phi$ . Notice that  $K_F^2 \neq 7$  by [Mo82]. We put  $F' = \psi(F)$  and  $F'' = \mu(F')$ . Its strict transform in  $X^+$  will be called  $\tilde{F}$ . Since we assume that  $X$  has index 1, (2.4) plus classification gives

$$(-K_X)^3 \leq 22.$$

The case that  $\phi$  is a  $\mathbb{P}_2$ -bundles was already treated in Part I ([JPR05]); here the only possible case is the small resolution of the quadric cone, which has index  $r_X = 3$ . This is already traeted in Propostion 2.12 above.

**From now on we shall assume that  $F \neq \mathbb{P}_2$  (for some or - equivalently - all fibers).**

**3.2. Notation.** (1) We introduce the number  $\lambda$  to be the maximal integer such that

$$H^0(-K_X - \lambda F) \neq 0.$$

(2) We recall the notations

$$(3.2.1) \quad \tilde{L}^+ = \alpha(-K_X) + \beta F$$



and

$$(3.2.2) \quad \tilde{F} = \alpha^+(-K_{X^+}) + \beta^+L^+.$$

If  $\dim Y^+ = 1$ , then we shall write  $F^+$  instead of  $L^+$ .

**3.3. Lemma.** *If  $K_F^2 = 8$ , then  $(-K_X)^3$  is divisible by 8.*

*Proof.* By [Mo82],  $X \subset \mathbb{P}(\mathcal{F})$  for some rank 4 bundle  $\pi : \mathcal{F} \rightarrow \mathbb{P}_1$ . Denote the tautological line bundle on  $\mathbb{P}(\mathcal{F})$  by  $\zeta$  and let  $X \in |2\zeta + \pi^*\mathcal{O}(\mu)|$  for some integer  $\mu$ . Then

$$-K_X = 2\zeta - \pi^*\mathcal{O}(c_1 + \mu - 2),$$

where  $c_1 = c_1(\mathcal{F})$ , i.e. there exists some integer  $b$  such that

$$L = \frac{1}{2}(-K_X + \pi^*\mathcal{O}(b)) \in \text{Pic}(X).$$

Now Riemann Roch for  $L$  gives

$$\chi(L) = 2 + 2b + \frac{(-K_X)^3}{8}$$

proving the claim.  $\square$

**3.4. Proposition.** *Consider the number  $\lambda$  introduced in (3.2). If  $K_F^2 < 8$ , all members of  $|-K_X - \lambda F|$  are irreducible and reduced.*

*Proof.* Let  $R \in |-K_X - \lambda F|$  and suppose that  $R = R_1 + R_2$ . Since  $K_F^2 < 8$ , the del Pezzo surface  $F$  contains  $(-1)$ -curves, hence say  $R_2|F \equiv 0$  (recall that  $R_i|F$  are proportional since  $\rho(X/Y) = 1$ ), so that  $R_2 = \phi^*(\mathcal{O}(a))$ , contradicting the maximality of  $\lambda$ .  $\square$

**3.5. Proposition.** *In the notations (3.2) the following holds.*

- (1)  $\beta\beta^+ = 1$ ,  $\alpha + \beta\alpha^+ = \alpha^+ + \beta^+\alpha = 0$ .
- (2) If  $K_F^2 \leq 6$  and if there exists a rational curve  $l^+ \subset X^+$  with  $-K_{X^+} \cdot l^+ = 1$  and  $L^+ \cdot l^+ = 0$ , then  $\beta = \beta^+ = -1$  and  $\alpha = \alpha^+ \in \mathbb{N}$ .
- (3) If  $K_F^2 \leq 6$  and a curve  $l^+$  as in (2) does not exist, then either (2) holds or  $(\beta, \beta^+) = (-2, -\frac{1}{2})$ .
- (4) If  $K_F^2 = 8$ , then either (2) holds or  $(\beta, \beta^+) = (-\frac{1}{2}, -2), (-2, -\frac{1}{2})$ .
- (5) Let  $D \subset X$  be an irreducible effective divisor with strict transform  $\tilde{D} \subset X^+$ . Then

$$K_X^2 \cdot D = K_{X^+}^2 \cdot \tilde{D}$$

and

$$K_X \cdot D^2 = K_{X^+} \cdot \tilde{D}^2.$$

*Proof.* (1) follows by inserting (3.2.1) into (3.2.2) and vice versa and by comparing coefficients.

(2) In the decomposition

$$\tilde{L}^+ = \alpha(-K_X) + \beta F, \tag{+}$$

the numbers  $\alpha, \beta$  are rational a priori. Suppose that  $K_F^2 \leq 6$ . Let  $l_\phi$  be a  $(-1)$ -curve in  $F$  and intersect with (+):

$$\tilde{L}^+ \cdot l_\phi = \alpha + 0,$$

hence  $\alpha \in \mathbb{N}$ . Thus  $\beta F$  is Cartier so that  $\beta \in \mathbb{Z}$ . Similarly, the existence of  $l^+$  gives  $\alpha \in \mathbb{N}$  and  $\beta^+ \in \mathbb{Z}$ . Since  $\beta < 0$ , the claim (2) follows.

(3) and (4) are done in the same way. We just observe that if there is no curve  $l^+$  with  $-K_{X^+} \cdot l^+ = 1$ , then at least we can find  $l^+$  such that  $-K_{X^+} \cdot l^+ = 2$ .

(5) Finally for (5) just represent the spanned line bundle  $-K_X$  by general members not meeting the exceptional locus of  $\psi$ .  $\square$

**3.6. Proposition.** *If  $\lambda = 0$ , then  $\beta \neq -\frac{1}{2}$ .*

*Proof.* Assume  $\lambda = 0$  and  $\beta = -\frac{1}{2}$ . Observe that  $\alpha \notin \mathbb{N}$ , because otherwise  $F$  would be divisible in  $\text{Pic}(X)$ . Hence we find a line bundle  $M$  such that

$$-K_X = F + 2M.$$

Since  $\lambda = 0$  we have  $H^0(-K_X - F) = 0$ , and therefore  $\phi_*(-K_X)$  has the form

$$\phi_*(-K_X) = \mathcal{O}^a \oplus \mathcal{O}(-1)^b.$$

Since  $a + b = K_F^2 + 1 = 9$  and  $a = h^0(-K_X) = \frac{(-K_X)^3}{2} + 3$ , we obtain  $(-K_X)^3 = 2a - 6$ , and we must have

$$(-K_X)^3 \leq 12.$$

The line bundle  $M + F = \frac{1}{2}(-K_X + F)$  is ample and by Kodaira vanishing and Riemann-Roch we obtain

$$h^0(M + F) = \chi(M + F) = \frac{(-K_X)^3}{8} + 4.$$

Hence  $(-K_X)^3 = 8$  and  $h^0(M + F) = 5$ . Consider the exact sequence

$$0 \rightarrow H^0(M) \rightarrow H^0(M + F) \rightarrow H^0((M + F)|F) = H^0(M|F).$$

Since  $\lambda = 0$ , we have  $H^0(M) = 0$ ; moreover  $M|F = \frac{-K_F}{2}$ , so that  $h^0((M + F)|F) = 4$ . This contradicts the exact sequence.  $\square$

**3.7. Proposition.** *Suppose that  $\dim Y^+ = 1$  and that  $\beta = -1$ . Then  $\alpha(-K_X)^3 = 2K_F^2$ .*

*Proof.* This is a consequence of  $K_X \cdot (\tilde{F}^+)^2 = 0$  and (3.5).  $\square$

First we consider the “exotic” cases in 3.5.

**3.8. Proposition.** *Assume  $\dim Y^+ = 1$  (and that  $X^+$  has index 1). Suppose  $K_F^2 = 8$ . Then  $\beta = -1$  except  $K_{F^+}^2 = 4$ ,  $(-K_X)^3 = 16$ , and  $(\alpha^+, \beta^+) = (1, -2)$ . This case really exists and  $X^+ \subset \phi_*^+(-K_{X^+}) = \mathbb{P}(\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O})$  may be realized as a complete intersection of two general sections in  $|2\zeta - 2F^+|$ .*

*Proof.* We may assume that  $\beta = -\frac{1}{2}$ ; the other case follows by interchanging the roles of  $X$  and  $X^+$ . Recall that  $\alpha \notin \mathbb{N}$ , because otherwise  $F$  would be divisible in  $\text{Pic}(X)$ . Hence we find a line bundle  $M$  such that

$$-K_X = F + 2M.$$

By (3.?),  $\lambda \geq 1$ . By cubing the equation  $-K_X - F = 2M$ , we obtain  $(-K_X)^3 = 8, 16$  with  $M^3 = -2, -1$ . Then Riemann-Roch gives  $\chi(M) \geq 1$ . Since  $h^q(M) = 0$  for  $q \geq 2$  (apply the Leray spectral sequence to  $\phi : X \rightarrow Y = \mathbb{P}_1$ ), we have

$$h^0(M) \geq 1.$$

Recall

$$\tilde{F}^+ = \alpha(-K_X) - \frac{1}{2}F$$

with  $\alpha$  a half-integer. Thus  $2\tilde{F}^+ = 2\alpha(-K_X) - F$ , and from  $h^0(2\tilde{F}^+) = 3$  and  $\lambda \geq 1$  we deduce  $\alpha = \frac{1}{2}$ . We obtain  $M = \tilde{F}^+$  and hence  $\lambda^+ = 2$ . The equation  $K_X \cdot (\tilde{F}^+)^2 = 0$  gives

$$\alpha(-K_X)^3 = K_F^2 = 8,$$

so that  $(-K_X)^3 = 16$ . Dually,  $K_{X^+} \cdot (\tilde{F})^2 = 0$  yields

$$\alpha^+(-K_X)^3 = 4K_{F^+}^2,$$

and with  $\alpha^+ = 2\alpha$ , we get  $K_{F^+}^2 = 4$ . This case in fact exists: First note  $\lambda^+ = 2$  and  $h^0(X^+, -K_{X^+} - 2F^+) = h^0(X, F) = 2$ . Let

$$\mathcal{E} = \phi_*^+(-K_{X^+}) = \mathcal{O}(2)^2 \oplus \mathcal{O}(1)^a + \mathcal{O}^b + \mathcal{O}^c.$$

Then  $h^0(-K_{X^+}) = 11$  and  $K_{F^+}^2 = 4$  gives  $2a + b = 5$  and  $a + b + c = 3$ , hence  $a = 2, b = 1$  and  $c = 0$ . Take  $X^+ \subset \mathbb{P}(\mathcal{E})$  a complete intersection of two sections

$$Q_i \in |\zeta - 2F^+|, \quad i = 1, 2$$

where the fiber of  $\mathbb{P}(\mathcal{E})$  is denoted by  $F^+$  as well. For  $Q_i$  general then  $X^+$  is smooth with  $-K_{X^+} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{X^+}$  is big and nef. Moreover, the exceptional curve  $C_0$  corresponding to the only trivial summand of  $\mathcal{E}$  is contained in  $Q_1$  and  $Q_2$ , hence  $C_0 \subset X^+$  is the exceptional locus of  $\psi^+$ . By construction,  $X^+$  admits a del Pezzo fibration with  $K_{F^+}^2 = 4$ .

Concerning the flop note first that  $N_{C_0/X^+}$  is of type  $(-1, -1)$ , hence  $X^+ \dashrightarrow X$  is a simple flop. The linear system

$$|\zeta - 2F^+|$$

on  $\mathbb{P}(\mathcal{E})$  defines a rational map onto  $\mathbb{P}_1$  with base locus a threefold  $Z$  containing  $C_0$ . It is easy to see that  $Z \cap Q_1 \cap Q_2 = C_0$ , hence  $X$  admits a del Pezzo fibration with  $K_F^2 = (-K_X - 2F^+) \cdot K_{X^+}^2 = 8$ .  $\square$

**3.9. Proposition.** *Suppose  $\dim Y^+ = 1$ . Then either*

- (1)  $\lambda = 1$  and  $((-K_X)^3, K_F^2) = (2, 1), (4, 2), (6, 3), (10, 5), (12, 6)$ , where the first three cases definitely exist; or
- (2)  $\lambda = 0$  and  $((-K_X)^3, K_F^2) = (2, 3), (2, 4), (2, 5), (2, 6), (4, 4), (4, 6), (6, 6)$ , where the cases  $(2, 3), (2, 4)$  and  $(4, 4)$  definitely exist.

*The existence of the remaining cases is open.*

*Proof.* By Prop. 3.8 we may assume  $\beta = \beta^+ = -1$  and  $\alpha = \alpha^+ \in \mathbb{N}$  in (3.5). So

$$\tilde{F}^+ = \alpha(-K_X) - F.$$

Since all members of  $|\tilde{F}^+|$  are irreducible, we must have  $\lambda \leq 1$  and if  $\lambda = 1$ , then  $\alpha = 1$ .

**Case I:**  $\lambda = 1$ .

Then  $-K_X = F + \tilde{F}^+$  and we obtain

$$\mathcal{E} = \phi_*(-K_X) = \mathcal{O}(1)^2 \oplus \mathcal{O}^b \oplus \mathcal{O}(-1)^c$$

with  $2 + b + c = K_F^2 + 1$  and  $4 + b = h^0(-K_X)$ . Since  $(-K_X)^3 = 2K_F^2$  by (3.7), it follows  $c = 0$  so that

$$H^1(-K_X - F) = H^1(\tilde{F}^+) = 0.$$

Thus

$$((-K_X)^3, K_F^2) = (2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (16, 8).$$

We will continue case by case.

**1.)**  $(-K_X)^3 = 2$ ,  $K_F^2 = 1$ . Here  $|-K_X|$  has base points; this case exists and was already treated in Proposition 2.5.

**2.)**  $(-K_X)^3 = 4$ ,  $K_F^2 = 2$ . This case exists, here  $X \rightarrow \mathbb{P}(\mathcal{E})$  with  $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)^2$  is a double covering, i.e.  $X$  is hyperelliptic. The construction can be found in Lemma 2.10.

**3.)**  $(-K_X)^3 = 6$ ,  $K_F^2 = 3$ . We construct this case as follows. Let  $\mathcal{E} = \mathcal{O}^2 \oplus \mathcal{O}(1)^2$  and define

$$X \in |3\zeta|$$

general. Then  $-K_X = \zeta|_X$ , hence  $X$  is a smooth almost Fano threefold with  $(-K_X)^3 = 6$ . The general fiber of the induced map  $X \rightarrow \mathbb{P}_1$  is a cubic in  $\mathbb{P}_3$ , hence a del Pezzo surface of degree 3. The map

$$\nu : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}_5$$

given by  $|\zeta|$  contracts the surface  $S \simeq \mathbb{P}_1 \times \mathbb{P}_1$  corresponding to the two trivial summands of  $\mathcal{E}$  along a ruling to a line  $C \subset \mathbb{P}_5$ . Since  $X \in |\nu^* \mathcal{O}_{\mathbb{P}_5}(3)|$  by construction,  $X \cap S$  consists of 3 fibers of  $\nu$ , i.e. the anticanonical map  $\psi : X \rightarrow X'$  contracts three smooth rational curves to points.

Note that the linear system  $|-K_X - F|$  is a pencil with base locus the exceptional locus of  $\psi$ . This shows  $X^+$  again admits a del Pezzo fibration.

**4.)**  $(-K_X)^3 = 8$ ,  $K_F^2 = 4$ . The construction is analogously to the last case: define  $X \subset \mathbb{P}(\mathcal{E})$  with  $\mathcal{E} = \mathcal{O}^3 \oplus \mathcal{O}(1)^2$  as a complete intersection of two general elements in  $|2\zeta|$ . Then  $X$  admits a del Pezzo fibration with  $K_F^2 = 4$  and the anticanonical map contracts 4 smooth rational curves. The flop is of the same type as above.

**5.)**  $(-K_X)^3 = 10$ ,  $K_F^2 = 5$ . Open.

**6.)**  $(-K_X)^3 = 12$ ,  $K_F^2 = 6$ . Open.

**7.)**  $(-K_X)^3 = 16$ ,  $K_F^2 = 8$ . This case does not exist for the following reason. By Proposition 2.4,  $X'$  admits a smoothing  $\mathcal{X}$  such that  $\mathcal{X}_t$  is a smooth Fano threefold of index 1 with  $(-K_{\mathcal{X}_t})^3 = 16$ . Then  $\mathcal{X}_t$  contains lines  $l_t$  and the degeneration of  $l_t$  to  $X$  gives a line  $l_0$  in  $X$ . Let  $\hat{l}_0 \subset X$  be the strict transform of  $l_0$ . Then

$$-K_X \cdot \hat{l}_0 = 1.$$

On the other hand,  $-K_X = F + \tilde{F}^+$  by assumption. Since  $-K_X|_F = K_F$  is divisible by two,  $\hat{l}_0$  cannot be contained in a fiber. This shows  $F \cdot \hat{l}_0 > 0$ . Then  $\tilde{F}^+ \cdot \hat{l}_0 \leq 0$ . But  $\hat{l}_0$  is not  $\psi$ -exceptional, hence  $\tilde{F}^+ \cdot \hat{l}_0 = 0$ . Then the strict transform of  $\hat{l}_0$  in  $X^+$  is contained in the fiber  $F^+$ , which is impossible as above.

**Case II:**  $\lambda = 0$ .

So  $\alpha(-K_X) = F + \tilde{F}^+$  with  $\alpha \geq 2$ . Here  $\mathcal{E} = \phi_*(-K_X)$  has the form

$$\mathcal{E} = \mathcal{O}^a \oplus \mathcal{O}(-1)^b.$$

Since  $a + b = K_F^2 + 1$  and  $a = h^0(-K_X) = \frac{(-K_X)^3}{2} + 3$ , we obtain

$$(-K_X)^3 = 2a - 6.$$

Thus  $a \geq 4$ . Now  $\alpha(-K_X)^3 = 2K_F^2$  hence only the following cases remain.

- (1)  $a = 4, (-K_X)^3 = 2$  and  $(K_F^2, \alpha) = (3, 3), (4, 4), (5, 5), (6, 6), (8, 8)$ ;
- (2)  $a = 5, (-K_X)^3 = 4$  and  $(K_F^2, \alpha) = (4, 2), (6, 3), (8, 4)$ ;
- (3)  $a = 6, (-K_X)^3 = 6$  and  $(K_F^2, \alpha) = (6, 2)$ ;
- (4)  $a = 7, (-K_X)^3 = 8$  and  $(K_F^2, \alpha) = (8, 2)$ .

We will continue case by case.

**1.)**  $a = 4, K_F^2 = 3$ . We have  $a = 4$  and  $b = 0$ , hence take

$$X \subset \mathbb{P}(\mathcal{O}^{\oplus 4}) \simeq \mathbb{P}_3 \times \mathbb{P}_1$$

a general element in  $|3\zeta + 2F| = |(3, 2)|$ . Then  $X$  is a smooth almost Fano threefold with the expected numerical data by adjunction formula. The anticanonical map is the restriction of the second projection

$$\pi : \mathbb{P}(\mathcal{O}^{\oplus 4}) \longrightarrow \mathbb{P}_3$$

defined by  $|\zeta|$ . Since  $X \cdot l_\pi = 2$  for a general fiber of  $\pi$ , the restriction of  $\pi$  to  $X$  factors, i.e.,  $X$  is hyperelliptic with

$$X \xrightarrow{\psi} X' \xrightarrow{\mu} \mathbb{P}_3,$$

$\mu$  a double covering. It remains to show that  $\psi$  is small. Let  $x_0, \dots, x_3, y_0, y_1$  be homogeneous coordinates of the product  $\mathbb{P}_3 \times \mathbb{P}_1$ . Then  $X$  is defined by some  $f(x_0, \dots, x_3, y_0, y_1)$ , homogeneous of degree 3 in the  $x_i$  and of degree 2 in the  $y_i$ . Write

$$f = g_0 y_0^2 + g_1 y_0 y_1 + g_2 y_1^2, \quad g_i \in \mathbb{C}[x_0, \dots, x_3]$$

homogeneous of degree 3. Since  $X$  was taken general, the  $g_i$  are general. Then the fiber over some point  $a = [a_0 : \dots : a_3] \in \mathbb{P}_3$  is contained in  $X$ , iff  $g_0(a) = g_1(a) = g_2(a) = 0$ . This shows  $X$  contains fibers of  $\pi$  exactly over the complete intersection  $g_0 = g_1 = g_2 = 0$  in  $\mathbb{P}_3$ , which are 27 points. Hence  $\psi$  is small with 27 exceptional curves. Since  $X$  is hyperelliptic, the flop  $X^+$  is of the same type as  $X$ .

**2.)**  $a = 4, K_F^2 = 4$ . We have  $a = 4$  and  $b = 1$ , hence take

$$X \subset \mathbb{P}(\mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1))$$

a complete intersection of two general elements

$$Q_1 \in |2\zeta + F|, \quad Q_2 \in |2\zeta + 2F|.$$

The base locus  $C_0 = \text{Bs}|\zeta|$  has  $C_0 \cdot Q_2 = 0$ . But the system  $|2\zeta + 2F|$  is base point free, hence the general element  $Q_2$  does not contain  $C_0$ . This shows  $X$  is a smooth almost Fano threefold with the expected numerical data. It remains to show  $\psi : X \rightarrow X'$  is small. This can be done either directly in coordinates as in the last case, or just by checking the respective lists for the divisorial case in [JPR05].

**3.)**  $a = 4, K_F^2 = 5$ . Open.

**4.)**  $a = 4, K_F^2 = 6$ . Open.

**5.)**  $a = 4, K_F^2 = 8$ . Then  $(-K_X)^3 = 2$ , which is impossible due to Lemma 3.3.

**6.)**  $a = 5, K_F^2 = 4$ . Here  $b = 0$ , hence

$$X \subset \mathbb{P}(\mathcal{O}^{\oplus 5}) \simeq \mathbb{P}_4 \times \mathbb{P}_1$$

has codimension 2. Since  $(-K_X)^3 = 4$ , either  $X' \subset \mathbb{P}_4$  is a quartic, or a double covering of a quadric, i.e. hyperelliptic. Both cases do exist.

(a) Take for  $X$  the complete intersection of two general elements

$$Q_0, Q_1 \in |2\zeta + F| = |(2, 1)|.$$

Then  $X$  is smooth almost Fano. Let  $x_0, \dots, x_4, y_0, y_1$  be homogeneous coordinates of  $\mathbb{P}_4 \times \mathbb{P}_1$ . Then  $Q_i$  is defined by

$$y_0 q_{i0} + y_1 q_{i1} = 0, \quad q_{ij} \in \mathbb{C}[x_0, \dots, x_4]$$

general quadrics. The image of  $\psi : X \rightarrow \mathbb{P}_4$  is given by the determinant of

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix},$$

hence a quartic in  $\mathbb{P}_4$ . We have exactly one point in  $X$  over any  $p \in \mathbb{P}_4$  with  $\text{rk} Q(p) = 1$ , and a whole  $\mathbb{P}_1$  over all  $p$  with  $\text{rk} Q(p) = 0$ . But  $\text{rk} Q(p) = 0$  means  $q_{ij}(p) = 0$  for all  $i, j$ , hence  $X \rightarrow X'$  has exceptional fibers over the intersection of the 4 general quadrics  $q_{ij}$  in  $\mathbb{P}_4$  cutting out 16 points. This shows  $\psi$  is small with 16 exceptional fibers.

Concerning the flop we consider the linear system  $|-2K_X - F| = |(2, -1)|_X|$  with base locus exactly  $\text{exc}(\psi)$ . Chasing successively the twisted ideal sequences of  $X \subset Q_0 \subset \mathbb{P}_4 \times \mathbb{P}_1$  we find  $h^0(X, (2, -1)|_X) = 2$ , i.e. the flop  $X^+$  again admits a del Pezzo fibration and is in fact of the same type as  $X$ .

(b) Take for  $X$  the complete intersection of

$$Q_0 \in |2\zeta| = |(2, 0)|, \quad Q_1 \in |2\zeta + 2F| = |(2, 2)|.$$

As above, the  $Q_i$  are given by

$$q_0 = 0, \quad y_0^2 q_1 + y_0 y_1 q_2 + y_1^2 q_3 = 0,$$

respectively. Now the image of  $X$  in  $\mathbb{P}_4$  is the quadric  $Q_0$ , the general fiber consists of two points, and we have again 16 exceptional fibers. Since  $X$  is hyperelliptic, the flop  $X^+$  is of the same type as  $X$ .

**7.)**  $a = 5$ ,  $K_F^2 = 6$ . Open.

**8.)**  $a = 5$ ,  $K_F^2 = 8$ . Here  $(-K_X)^3 = 4$ , contradicting Lemma 3.3.

**9.)**  $a = 6$ ,  $K_F^2 = 6$ . Open.

**10.)**  $a = 7$ ,  $K_F^2 = 8$ . This case does not exist by the following argument. By [Mo82],  $X \subset \mathbb{P}(\mathcal{F})$  for some rank 4 vector bundle  $\mathcal{F}$  on  $\mathbb{P}_1$  and

$$X \in |2\zeta + \pi^* \mathcal{O}(\mu)|$$

for some integer  $\mu$ . By adjunction formula, we have

$$-K_X = 2\zeta + (2 - c_1 - \mu)F$$

with  $c_1 = c_1(\mathcal{F})$ . Assume  $\mathcal{F}$  is normalized such that  $-3 \leq c_1 \leq 0$ . Then

$$8 = (-K_X)^3 = -8c_1 - 16\mu + 48,$$

hence  $-K_X = 2\zeta + (-\frac{c_1}{2} - \frac{1}{2})F$ . Since  $-K_X$  is a line bundle and not divisible in  $\text{Pic}(X)$  by assumption, we must have  $c_1 = -3$  and  $\mu = 4$ .

Now  $\alpha = 2$  gives  $\tilde{F}^+ = -2K_X - F$ , i.e.

$$\tilde{F}^+ = 4\zeta|_X + F.$$

Hence  $h^0(X, \zeta|_X) = 0$ . The twisted ideal sequence of  $X$  shows  $\mathcal{F} = \pi_*\zeta = \pi_*(\zeta|_X)$ , i.e.

$$H^0(\mathbb{P}_1, \mathcal{F}) = 0.$$

Assume  $\mathcal{E} = \bigoplus_{i=1}^4 \mathcal{O}(a_i)$ . Then  $\sum a_i = -3$  and  $a_i < 0$  for all  $i$ . This is impossible.  $\square$

**3.10. Proposition.** *Assume  $\dim Y^+ = 2$  and let  $\tau^+$  be the degree of the discriminant locus of  $\phi^+$ . Then, using the notations of (3.2),*

- (1) *Either  $(\beta, \beta^+) = (-1, -1)$  or  $(\beta, \beta^+) = (-\frac{1}{2}, -2)$ .*
- (2)  *$K_X \cdot (\tilde{L}^+)^2 = -2$ .*
- (3)  *$\alpha^2(-K_X)^3 = 2\alpha(12 - \tau^+) - 2$ .*
- (4)  *$\alpha^2(-K_X)^3 = 2\alpha K_F^2 + 2$  if  $\beta = -1$ ; otherwise  $\alpha^2(-K_X)^3 = \alpha K_F^2 + 2$ .*

*Proof.* (1) follows from (3.5): the case  $\phi^+$  is a  $\mathbb{P}_1$ -bundle is treated in section 4, we may hence assume  $\phi^+$  is a proper conic bundle. i.e. there are singular fibers and therefore there exists a curve  $l^+$  with  $-K_{X^+} \cdot l^+ = 1$ .

(2) This is (3.5), (5).

(3) is a consequence of (1) and (3.7), having in mind that  $(-K_X)^3 = (-K_{X^+})^3$ .

(4) is a consequence of (1) and (3).  $\square$

**3.11. Proposition.** *If  $\dim Y^+ = 2$ , then  $\beta = -1$  and  $((-K_X)^3, K_F^2, \tau^+)$  is one of  $(8, 3, 7)$ ,  $(10, 4, 6)$ ,  $(12, 5, 5)$ ,  $(14, 6, 4)$ , where the first two cases really exist.*

*Proof.* (1) We first consider the case that  $\beta = -1$ . Then 3.10(3) and (4) give

$$\alpha(12 - \tau^+ - K_F^2) = 2.$$

Hence either

$$\alpha = 1; \quad 12 - \tau^+ - K_F^2 = 2,$$

or

$$\alpha = 2; \quad 12 - \tau^+ - K_F^2 = 1.$$

The second alternative however contradicts 3.10(4). So  $\alpha = 1$ . Here  $(-K_X)^3 = 2K_F^2 + 2$  and  $-K_X = F + \tilde{L}^+$ . So  $h^0(-K_X - F) = 3$  and  $|-K_X - F|$  does not contain reducible members. This implies

$$\mathcal{E} = \phi_*(-K_X) = \mathcal{O}(1)^3 \oplus \mathcal{O}^b \oplus \mathcal{O}(-1)^c$$

with  $3 + b + c = K_F^2 + 1$  and  $h^0(-K_X) = 6 + b$ . Hence  $c = 0$  and

$$K_F^2 = b + 2; \quad (-K_X)^3 = 2b + 6.$$

Next we consider the spanned rank 3 bundle

$$\mathcal{E}^+ = \phi_*^+(-K_{X^+}).$$

Using the notations of section 4,  $X^+ \subset \mathbb{P}(\mathcal{E}^+)$  is a divisor of the form

$$[X] = 2\zeta + \pi^*(\mathcal{O}(\lambda)), \quad \lambda = 3 - c_1,$$

such that  $\zeta|_{X^+} = -K_{X^+}$ . Let  $c_i = c_i(\mathcal{E}^+)$ . Then

$$2b + 6 = (-K_X)^3 = \zeta^3 \cdot X = c_1^2 - 2c_2 + 3c_1. \quad (*)$$

The equation  $K_{X^+}^2 \cdot L^+ = 12 - \tau^+$  translates into

$$c_1 = 9 - \tau^+. \quad (**)$$

From  $(-K_X)^3 = 2b + 6$  and 3.10(3) we get

$$b = 8 - \tau^+,$$

in particular  $2 \leq \tau^+ \leq 8$ ,  $\tau^+ \neq 3$ . Putting this and  $(**)$  into  $(*)$  gives

$$(\tau^+)^2 - 19\tau^+ + 86 = 2c_2.$$

We consider a general section in  $\mathcal{E}^+$  and obtain a rank 2 vector bundle  $\mathcal{F}^+$  from the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^+ \rightarrow \mathcal{F}^+ \rightarrow 0.$$

Hence  $h^0(\mathcal{F}^+(-1)) = h^0(\mathcal{E}^+(-1)) = 2$ . We continue case by case.

1.)  $\tau^+ = 8$ . Then  $c_1(\mathcal{F}^+) = 1$ ,  $c_2(\mathcal{F}^+) = -1$ , which is impossible.

2.)  $\tau^+ = 7$ . Then  $K_F^2 = 3$  and

$$X \subset \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O})$$

is a hypersurface. Take  $X \in |3\zeta - F|$  general. Then  $X$  is a smooth almost Fano threefold with  $-K_X = \zeta|_X$ . The map

$$\mathbb{P}(\mathcal{E}) \xrightarrow{|\zeta|} Z \subset \mathbb{P}_6$$

is a small resolution of the cone over  $\mathbb{P}_1 \times \mathbb{P}_2 \hookrightarrow \mathbb{P}_5$  (Segre embedding). The exceptional curve  $C_0 \subset \mathbb{P}(\mathcal{E})$  corresponds to the projection  $\mathcal{E} \rightarrow \mathcal{O}$ , the only trivial summand of  $\mathcal{E}$ . Since

$$X \cdot C_0 = (3\zeta - F) \cdot C_0 = -1,$$

$X$  contains  $C_0$ , hence  $\psi$  is small with exactly one exceptional curve.

Concerning the flop note that the normal bundle of  $C_0$  in  $X$  is of type  $(-1, -1)$ , i.e.  $X \dashrightarrow X^+$  is a simple flop. The linear system

$$|(\zeta - F)|_X|$$

has base locus exactly  $C_0$  and we find  $h^0(X, \zeta - F) = 3$ . This shows  $X^+$  admits a conic bundle structure and the remaining data are easily verified.

3.)  $\tau^+ = 6$ . Then  $K_F^2 = 4$  and

$$X \subset \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}^{\oplus 2}),$$

a small resolution of the double cone  $Z \subset \mathbb{P}_7$  over  $\mathbb{P}_1 \times \mathbb{P}_2 \hookrightarrow \mathbb{P}_5$  embedded by Segre, i.e.

$$\mathbb{P}(\mathcal{E}) \xrightarrow{|\zeta|} Z \subset \mathbb{P}_7.$$

Take  $X$  a complete intersection of

$$Q_1 \in |2\zeta|, \quad Q_2 \in |2\zeta - F|$$

general. Then  $X$  is a smooth almost Fano threefold with  $-K_X = \zeta|_X$ . The first quadric  $Q_1$  is the pullback of some general quadric in  $\mathbb{P}_7$  intersecting the vertex of the cone  $Z$  in two points. This means

$$Q_1 \longrightarrow Z_1 \subset \mathbb{P}_7$$



is birational with exceptional locus two smooth rational curves  $C_1, C_2 \subset Q_1$ . The threefold  $X$  is a divisor on  $Q_1$ , cut out by  $Q_2$ , and we find

$$X \cdot (C_1 + C_2) = Q_1 \cdot Q_2 \cdot S = -2,$$

where  $S \simeq \mathbb{P}_1 \times \mathbb{P}_1 = \mathbb{P}(\mathcal{O}^{\oplus 2})$  is the exceptional surface of  $\mathbb{P}(\mathcal{E}) \rightarrow Z$ . Since the two curves are numerically equivalent, both have negative intersection number with  $X$  in  $Q_1$ , are hence contained in  $X$ . This shows  $\psi$  is small with two smooth exceptional curves.

Concerning the flop consider as above the linear system  $|\zeta - F|$  on  $X$ , which has base locus  $C_1 \cup C_2$  on  $X$  and admits 3 sections. The remaining data of  $X^+$  now follow numerically.

4.)  $\tau^+ = 5$ . Open.

5.)  $\tau^+ = 4$ . Open.

6.)  $\tau^+ = 3$ . Then  $K_F^2 = 7$ , which is impossible.

7.)  $\tau^+ = 2$ . Then  $K_F^2 = 8$  and  $(-K_X)^3 = 18$ , contradicting Lemma 3.3.

8.)  $\tau^+ = 1$ . Then  $K_F^2 = 9$ , which is ruled out by assumption.

(2) Now consider the case  $\beta = -\frac{1}{2}$  so that  $\beta^+ = -2$  and  $K_F^2 = 8$ . Arguing in the same way as in (1), we get

$$\alpha(48 - 4\tau^+ - K_F^2) = 6.$$

Together with 3.10(4) this yields a contradiction.  $\square$

**3.12. Setup.** Assume that  $\phi^+$  is birational. The exceptional divisor will be denoted  $E^+$  and its strict transform in  $X$  by  $\tilde{E}^+$ . Slightly differing from (3.2), we will substitute  $L^+$  by  $E^+$  and shall write

$$(3.12.1) \quad \tilde{F} = \alpha^+(-K_{X^+}) + \beta^+E^+$$

and

$$(3.12.2) \quad \tilde{E}^+ = \alpha(-K_X) + \beta F.$$

All results of Proposition 3.5 remain valid. Denote the generator of  $\text{Pic}(Y^+)$  by  $H^+$ , i.e.,  $L^+ = (\phi^+)^*H^+$ . If  $Y^+$  is smooth, then let  $-K_{Y^+} = rH^+$ ,  $1 \leq r \leq 4$  the index of  $Y^+$ .

**3.13. Proposition.** *Suppose  $E^+ = \mathbb{P}_2$  with normal bundle  $\mathcal{O}(-1)$ . Then  $Y^+$  has index 1 and  $(-K_{Y^+})^3 = 18$ . Moreover  $(-K_X)^3 = 10$  and  $K_F^2 = 6$ .*

*Proof.* Suppose that  $\phi^+$  is the blow-up of the smooth point  $p$  in the Fano threefold  $Y^+$ . First notice that  $Y^+$  is a smooth Fano threefold with index 1. In fact, if  $Y^+$  had index 2, then  $X$  had index 2, which we ruled out. If  $Y^+$  had index 3 or 4, then  $X^+$  would be Fano. By intersecting (3.12.1) with a general line in  $Y^+$  not meeting  $\phi^+(E^+)$ , we get  $\alpha^+ \in \mathbb{N}$ , hence  $-\beta^+ \in \mathbb{N}$ . We also notice that

$$K_X \cdot (\tilde{E}^+)^2 = 2. \quad (*)$$

(1) Suppose that  $-\beta \in \mathbb{N}$ . Then  $\beta = \beta^+ = -1$ . Combining (\*) with (3.12.2) gives

$$\alpha(\alpha K_X^3 + 2K_F^2) = 2,$$

hence  $\alpha = 1, 2$ . If  $\alpha = 2$ , then  $2(-K_X)^3 = 2K_F^2 + 1$ , so that only  $\alpha = 1$  remains. Next we combine  $K_{X^+} \cdot \tilde{F}^2 = 0$  with (3.12.1), so that

$$\alpha^2(-K_X)^3 + 8\alpha = -2,$$

hence

$$(-K_X)^3 = 10, \quad K_F^2 = 6.$$

Conversely, this case really exists by [Ta89].

(2) If  $-\beta \notin \mathbb{N}$ , then  $\beta = -\frac{1}{2}$  and  $\beta^+ = -2$ . Here we obtain from  $\tilde{E}^+ = \frac{1}{2}\alpha^+(-K_X) - \frac{1}{2}F$  and (\*) that

$$\alpha^+(\alpha^+K_X^3 + 2K_F^2) = 8.$$

Since  $\alpha^+$  is an odd positive integer, we conclude  $\alpha^+ = 1$ . Then equation (3.12.1) gives  $(-K_X)^3 = 24$ , hence  $(-K_{Y^+})^3 = 32$ , which is impossible by classification.  $\square$

**3.14. Proposition.** *Suppose  $E^+$  is a quadric, either smooth or a quadric cone. Then  $\beta = \beta^+ = -1$ ,  $\alpha = \alpha^+ = 1$ ,  $(-K_X)^3 = 6$ ,  $K_F^2 = 4$ ,  $(-K_{Y^+})^3 = 8$  and  $X \subset \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}^4)$  may be realized as a complete intersection.*

*Proof.* By (2.7) and (2.10) we may assume that  $X'$  is not hyperelliptic. Since  $-K_{X^+}|E^+ = \mathcal{O}(1)$ , necessarily

$$H^0(-K_{X^+}) \rightarrow H^0(-K_{X^+}|E^+)$$

is surjective, since  $\psi^+|E^+$  must have degree 1. Thus  $\psi^+|E^+$  is actually an isomorphism. Consequently, if  $l^+$  is a curve contracted by  $\psi^+$ , then

$$E^+ \cdot l^+ = 1.$$

Since  $-K_{X^+} \cdot l^+ = 0$ , (3.12.1) implies  $-\beta^+ \in \mathbb{N}$ . Intersecting (3.12.1) with a line in  $E^+$  we furthermore see that  $\alpha^+ - \beta^+ \in \mathbb{N}$ , hence  $\alpha^+ \in \mathbb{N}$ .

(1) Suppose first that  $-\beta \in \mathbb{N}$ . Then (3.12.2) gives  $\beta = -1 = \beta^+$  and  $\tilde{F} \cdot l^+ = -1$ . It is also clear that  $\alpha^+ = 1$  since  $h^0(-K_{X^+} - E^+) \neq 0$ . So  $h^0(-K_X) = 6$ , so that  $(-K_X)^3 = 6$  and  $K_F^2 = 4$ ; furthermore  $(-K_{Y^+})^3 = 8$ .

This case really exists: We have  $h^0(-K_X - F) = 1$ , hence

$$X \subset \mathbb{P}(\phi_*(-K_X)) = \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}^{\oplus 4}) = \text{Bl}_{\mathbb{P}_3}(\mathbb{P}_5).$$

To realize  $X$  take a complete intersection of  $Z_1 \in |2\zeta|$  and  $Z_2 \in |2\zeta + F|$  general. Then  $X$  is smooth with  $-K_X = \zeta|_X$ . The anticanonical map  $\psi : X \rightarrow X'$  is induced by the map

$$p : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}_5$$

contracting  $D \simeq \mathbb{P}_1 \times \mathbb{P}_3$  to  $\mathbb{P}_3$ . The intersection  $E = D \cap X$  is a smooth surface, mapped by  $p$  birationally to a quadric in  $\mathbb{P}_3$  with exceptional locus 8 rational curves. After flopping these curves, the strict transform  $E^+$  of  $E$  becomes a contractible quadric.

(2) If  $-\beta \notin \mathbb{N}$ , then  $\beta = -\frac{1}{2}$  and  $\beta^+ = -2$ . Arguing as in 3.13, (2), we obtain

$$\alpha^+ = 1, K_X^3 + 2K_F^2 = 8.$$

On the other hand,  $K_{X^+} \cdot \tilde{F}^2 = 0$  leads to  $(-K_X)^3 = 16$ , so that  $K_F^2 = 12$ , which is absurd.  $\square$

**3.15. Proposition.** *Suppose  $E^+ = \mathbb{P}_2$  with normal bundle  $\mathcal{O}(-2)$ . Then  $X \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus 3}) = \mathbb{P}(\mathcal{E})$  is a smooth element of  $|3\zeta + \pi^*\mathcal{O}(1)|$ ; and the flop of  $X$  really has a contraction contracting a  $\mathbb{P}_2$  with normal bundle  $\mathcal{O}(-2)$ .*

*Proof.* The first part of the proof is parallel to the proof of 3.14. If  $-\beta \in \mathbb{N}$ , we end up with  $(-K_X)^3 = 4$  and  $K_F^2 = 3$  and necessarily  $X \subset \mathbb{P}(\phi_*(-K_X))$  with  $\phi_*(-K_X) = \mathcal{O}(1) \oplus \mathcal{O}^3$  over  $\mathbb{P}_1$  is in the linear system as stated in the proposition. If conversely - in the obvious notation -  $X \in |3\zeta + F|$  is a smooth element, then  $-K_X = \zeta|_X$  and it is easily checked that the blow-down

$$\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_4$$

defined by  $\zeta$  restrict to a small contraction on  $X$ . It remains to show that  $\phi^+$  really contracts a plane with normal bundle  $\mathcal{O}(-2)$ . This can be checked directly: let  $D \subset \mathbb{P}(\mathcal{E})$  be the exceptional divisor and  $E = X \cap D$ . Then  $E$  is a smooth surface and the projection map  $D \rightarrow \mathbb{P}_2$  restricts to a birational map on  $E$ , contracting 9 rational curves. These curves are the exceptional locus of  $\psi : X \rightarrow X'$  and are of type  $(-1, -1)$ . After flop, the strict transform  $E^+$  of  $E$  becomes a contractible  $\mathbb{P}_2$ . Following the restriction  $-K_X|_E$  through the flop diagram we find  $-K_{X^+}|_{E^+} = \mathcal{O}(1)$ , hence  $N_{E^+/X^+} = \mathcal{O}(-2)$  as claimed.

If  $-\beta \notin \mathbb{N}$ , then  $-\beta = \frac{1}{2}$  and by the same computations as in (3.14) we obtain  $\alpha^+ = 1$  and then that  $(-K_X)^3 = 12$  and hence  $K_F^2 = 10$  which is absurd.  $\square$

**3.16. Proposition.** *Suppose  $\dim \phi^+(E^+) = 1$ . Then  $X^+ = \text{Bl}_{C^+}(Y^+)$  the blowup of a smooth Fano threefold  $Y^+$  along a smooth curve  $C^+$  by [Mo82] and  $X$  is one of the cases listed in table A.4 in the appendix.*

*Proof.* (1) Let  $l^+$  be a curve contracted by  $\phi^+$ . Intersecting (3.12.1) with  $l^+$  gives

$$\alpha^+ - \beta^+ \in \mathbb{N}.$$

Choose a line  $C' \subset Y^+$ . If  $r = 1$ , we can choose  $C'$  disjoint from  $C^+$ . In fact, suppose that all lines meet  $C^+$  and consider their strict transforms  $C$  in  $X^+$ . Then  $K_{X^+} \cdot C = 0$  so that  $\psi^+$  would not be small. Hence the general line  $C'$  is disjoint from  $C^+$ . Then we intersect (3.12.1) with  $C'$  and obtain  $\alpha^+ \in \mathbb{N}$ , hence  $-\beta^+ \in \mathbb{N}$ . In the other cases we simply get

$$r\alpha^+ \in \mathbb{N},$$

hence  $r(-\beta)^+ \in \mathbb{N}$ , too.

(2) Now the reasoning of (3.5) applies: we have  $\beta\beta^+ = 1$  and  $\alpha + \alpha^+\beta = 0$ . If  $K_F^2 \leq 6$ , then  $\alpha, \beta \in \mathbb{Z}$ . The case  $\alpha, \beta \notin \mathbb{Z}$  may only happen if  $K_F^2 = 8$ , then  $\alpha = \frac{\tilde{\alpha}}{2}, \beta = \frac{\tilde{\beta}}{2}$  with  $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}$  odd. Define

$$\alpha', \beta' := \begin{cases} \alpha, \beta & \text{if } \alpha, \beta \in \mathbb{Z} \\ 2\alpha, 2\beta & \text{if } \alpha, \beta \notin \mathbb{Z} \end{cases}$$

Then  $r\beta^+ \in \mathbb{Z}$  implies

$$(3.16.1) \quad \beta' \mid r.$$

Let  $d$  resp.  $g$  be the degree resp. the genus of  $C^+ \subset X^+$ . Then the following formulas are well-known (see e.g. [JPR05], p.603).

$$\begin{aligned} (E^+)^3 &= -rd + 2 - 2g; \\ K_{X^+}^2 \cdot E^+ &= rd + 2 - 2g; \\ K_{X^+} \cdot (E^+)^2 &= 2 - 2g; \\ (-K_X)^3 &= r^3(L^+)^3 - 2rd + 2g - 2. \end{aligned}$$

Using these equations and introducing

$$\sigma := rd + 2 - 2g,$$

we find the following relations:

$$\begin{aligned} (1) \quad & 0 = \alpha^2 K_X^3 + 2\alpha\sigma + 2 - 2g, \\ (2) \quad & 0 = \alpha(-K_X)^3 + \beta K_F^2 - \sigma, \\ (3) \quad & 0 = \alpha^2 + 2\alpha\beta K_F^2 + 2 - 2g, \\ (4) \quad & \left( \frac{\alpha'(\alpha'+1)(2\alpha'+1)}{12} (-K_X)^3 + 2\alpha' + 1 \right) - 1 \leq -\beta' \left( \frac{\alpha'(\alpha'+1)}{2} K_F^2 + 1 \right). \end{aligned}$$

Here  $K_X \cdot F^2 = 0$ ,  $K_F^2 = K_X^2 \cdot F$  and  $\tilde{E}^+ = \alpha(-K_X) + \beta F$  imply (1) and (2). Equation (3) follows from (1) and (2). To show (4) consider the ideal sequence of  $-\beta' > 0$  general fibers on  $X$  and twist with  $\alpha'(-K_X)$ :

$$0 \longrightarrow \mathcal{O}_X(\epsilon \tilde{E}^+) \longrightarrow \mathcal{O}_X(-\alpha' K_X) \longrightarrow \bigoplus_{-\beta'} \mathcal{O}_F(-\alpha' K_F) \longrightarrow 0,$$

with  $\epsilon = 1, 2$  (depending on whether  $\beta' = \beta \in \mathbb{Z}$  or not). Now  $h^0(X, \epsilon \tilde{E}^+) = 1$  and Riemann-Roch on  $X$  and  $F$ , respectively, shows the claim.

To run a computer program we have to prove effective bounds for all data involved. Since  $-K_{X+|E^+}$  is still big and nef, we have  $\sigma > 0$ . Since  $\psi$  is small,  $X'$  has only terminal singularities, is hence smoothable by [Na97]. Then the smoothing  $\mathcal{X}_t$  has the same index as  $X$  by [JR06a], which is 1 by assumption. Moreover,  $|-K_X|$  is base point free, hence

$$4 \leq (-K_X)^3 \leq 22.$$

The image  $Y^+$  of  $\phi^+$  is a smooth Fano threefold of index  $r$ , i.e.  $1 \leq r \leq 4$  and

$$0 < (-K_{Y^+})^3 \leq \begin{cases} 22, & r = 1 \\ 40, & r = 2 \end{cases} \quad \text{and} \quad (-K_{Y^+})^3 = \begin{cases} 54, & r = 3 \\ 64, & r = 4 \end{cases}$$

Then  $22 \geq (-K_X)^3 = (-K_Y)^3 - \sigma - rd \geq 4$  gives

$$d \leq \frac{(-K_{Y^+})^3 - 5}{r} \leq 17 \quad \text{and} \quad \sigma \leq (-K_{Y^+})^3 - r - 4 \leq \begin{cases} 17, & r = 1 \\ 34, & r = 2 \\ 47, & r = 3 \\ 54, & r = 4 \end{cases}$$

Finally  $\sigma = rd - 2g + 2$  implies  $g \leq \frac{17r}{2} + 1$ .

Running a computer program (written in C), this leads to the following tabular:

Nr.	$r$	$(-K_X)^3$	$K_F^2$	$g$	$d$	$\alpha$	$\beta$	$(L^+)^3$
1	1	4	6	1	6	3	-1	16
2	1	6	6	1	6	2	-1	18
3	1	8	5	0	1	1	-1	12
4	1	10	6	0	2	1	-1	16
5	2	4	6	1	3	3	-1	2
6	2	10	6	0	1	1	-1	2
7	2	4	3	1	3	3	-2	2
8	2	4	5	1	5	5	-2	3
9	2	6	4	4	8	3	-2	4
10	2	8	2	1	2	1	-2	2
11	2	8	6	1	6	3	-2	4
12	2	10	3	0	1	1	-2	2
13	2	12	3	1	3	1	-2	3
14	2	14	4	0	2	1	-2	3
15	2	16	4	1	4	1	-2	4
16	2	18	5	0	3	1	-2	4
17	2	22	6	0	4	1	-2	5
18	3	4	5	9	11	8	-3	2
19	3	14	4	5	8	2	-3	2
20	3	16	5	3	7	2	-3	2
21	3	18	6	1	6	2	-3	2
22	3	8	8	8	10	$\frac{7}{2}$	$-\frac{3}{2}$	2
23	4	4	6	15	11	7	-2	1
24	4	4	3	15	11	7	-4	1
25	4	6	5	8	9	7	-4	1
26	4	10	3	10	9	3	-4	1
27	4	12	4	7	8	3	-4	1
28	4	14	5	4	7	3	-4	1
29	4	16	6	1	6	3	-4	1

(3) Assume  $\beta = r$ . Then  $r \mid (\alpha + 1)$  and we several times use the following argument: let  $F_1, F_2 \in |F|$  be two general elements. Then the strict transforms  $\tilde{F}_1^+, \tilde{F}_2^+ \in |\alpha L^+ - \frac{\alpha+1}{r} E^+|$  cut out the exceptional curves for  $\psi$  and we find the degree of the exceptional curves of  $\psi$  in  $Y^+$

$$(3.16.2) \quad L^+ \cdot \text{exc}(\psi) = L^+ \cdot (\alpha L^+ - \frac{\alpha+1}{r} E^+)^2 = \alpha^2 (L^+)^3 - (\frac{\alpha+1}{r})^2 d,$$

hence

$$(3.16.3) \quad \alpha^2 (L^+)^3 > (\frac{\alpha+1}{r})^2 d.$$

We now consider the cases 1-29 seperately.

(3a) Assume  $r = 1$ .

**No.1,2:** open.

**No.3,4:** exist by [Isk89] and [Ta89], respectively.

(3b) Assume  $r = 2$ .

**No.5,6:** Here  $F$  would be divisible, i.e. they cannot exist.

**No.7:** We have  $\tilde{F}^+ = -K_{X^+} + (L^+ - E^+)$ , hence  $h^0(X^+, L^+ - E^+) = 0$ . Consider the twisted ideal sequence of  $C^+$  in  $Y^+$

$$0 \longrightarrow \mathcal{I}_{C^+}(H^+) \longrightarrow \mathcal{O}_{Y^+}(H^+) \longrightarrow \mathcal{O}_{C^+}(H^+) \longrightarrow 0.$$

By Riemann-Roch on  $Y^+$  we have  $h^0(Y^+, H^+) = (H^+)^3 + 2 = 4$ . Riemann-Roch on  $C^+$  gives  $h^0(C^+, H^+|_{C^+}) = 3$ . Then  $h^0(Y^+, \mathcal{I}_{C^+}(H^+)) \neq 0$ , a contradiction, so this case does not exist.

**No.8:** Open.

**No.9:** Does not exist by the same argument as in No.7.

**No.10:** Excluded by (3.16.3).

**No.11:** Open.

**No.12:** This case exists, we will give a construction. Here  $\tilde{F}^+ = L^+ - E^+$  and (3.16.2) shows  $\phi^+(\text{exc}(\psi))$  is a line. For the construction, we assume the image in  $Y^+$  of a general  $\tilde{F}_1^+$  is a smooth surface  $S \in |H^+|$ . Then the restriction of another  $\tilde{F}_2^+$  to  $S$  splits into  $C^+$  and the exceptional locus  $R$  of  $\psi$ . We construct  $R$ ,  $C^+$ ,  $S$  and  $Y^+$  explicitly:

Let  $\nu : Y^+ \rightarrow \mathbb{P}_3$  be a double covering ramified along a general quartic, i.e.  $Y^+$  is a smooth Fano threefold with  $-K_{Y^+} = \nu^*\mathcal{O}_{\mathbb{P}_3}(2) = 2H^+$ . Take  $S \subset Y^+$  a general element in  $|H^+|$ . Then  $H^+$  restricts to  $-K_S$  and  $S$  is a smooth del Pezzo surface of degree 2. Hence

$$\pi : S \rightarrow \mathbb{P}_2$$

may be realized as blowup in 7 points in general position. Let  $l_1, \dots, l_7$  be the exceptional curves for  $\pi$ . Define

$$R \in |\pi^*\mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2|, \quad \text{and} \quad C^+ \in |\pi^*\mathcal{O}_{\mathbb{P}_2}(2) - l_2 - \dots - l_7|$$

general. Then  $R + C^+ = -K_S$ ,  $R \cdot C^+ = 2$ , and  $R, C^+$  are both lines in  $Y^+$ . Define

$$X^+ = \text{Bl}_{C^+}(Y^+).$$

To show  $-K_{X^+}$  is nef it is enough to prove  $|\mathcal{I}_{C^+}(2H^+)|$  is base point free. Consider the twisted ideal sequence

$$0 \longrightarrow \mathcal{I}_S(2H^+) \longrightarrow \mathcal{I}_{C^+}(2H^+) \longrightarrow \mathcal{I}_{C^+/S}(2H^+) \longrightarrow 0.$$

Then all sections in  $H^0(S, \mathcal{I}_{C^+/S}(2H^+))$  lift to  $Y^+$  since  $H^1(Y^+, \mathcal{I}_S(2H^+)) = H^1(Y^+, \mathcal{O}_{Y^+}(H^+)) = 0$ . This means it suffices to prove  $|\mathcal{I}_{C^+/S} \otimes \mathcal{O}_{Y^+}(2H^+)|_S|$  is base point free. We have  $H^+|_S = -K_S$  and

$$2H^+|_S - C^+ = (\pi^*\mathcal{O}_{\mathbb{P}_2}(2) - l_1 - l_2 - l_3 - l_4) + (\pi^*\mathcal{O}_{\mathbb{P}_2}(2) - l_1 - l_5 - l_6 - l_7),$$

hence the sum of two systems of quadrics in  $\mathbb{P}_2$  through 4 general points. These are base point free. Just numerically we find  $(-K_X)^3 = 10 > 0$  as claimed.

Since  $C^+$  meets the line  $R$  in 2 points transversally, the strict transform  $R^+$  of  $R$  in  $X^+$  is a smooth anticanonically trivial rational curve, hence contracted by  $\psi^+$ . It remains to show the flop  $X$  exists and admits a del Pezzo fibration. First the normal bundle of  $R^+$  in  $X^+$  is of type  $(-1, -1)$ : let  $S^+ \simeq S$  be the strict transform of  $S$  in  $X^+$ . Then  $N_{R^+/S^+} = \mathcal{O}(-1)$  and we have

$$0 \longrightarrow N_{R^+/S^+} \longrightarrow N_{R^+/X^+} \longrightarrow N_{S^+/X^+}|_{R^+} \longrightarrow 0.$$

Since  $R^+$  is anticanonically trivial, the degree of  $N_{R^+/X^+}$  is 2, hence  $N_{S^+/X^+}|_{R^+} = \mathcal{O}(-1)$  and the sequence splits. This shows the flop is a simple flop.

By assumption, the pencil on  $X$  should be given by the strict transform of the system  $|L^+ - E^+|$ . The twisted ideal sequence

$$0 \longrightarrow \mathcal{I}_S(H^+) \longrightarrow \mathcal{I}_{C^+}(H^+) \longrightarrow \mathcal{I}_{C^+/S}(H^+) \longrightarrow 0$$

shows  $h^0(Y^+, \mathcal{I}_{C^+}(H^+)) = 1 + h^0(S, \mathcal{I}_{C^+/S}(H^+)) = 2$  and the base locus is exactly  $R$ . This gives a map  $X \rightarrow \mathbb{P}_1$  and  $K_F^2 = 3$  follows easily.

**No.13:** Excluded by (3.16.3).

**No.14:** This case exists and can be constructed the same way as No.12 above. Here  $Y^+ \subset \mathbb{P}_4$  is a cubic, hence a general  $S \in |H^+|$  is a cubic surface, i.e. the blowup of  $\mathbb{P}_2$  in 6 points. With the same notation as in No.12, define

$$R \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2| \quad \text{and} \quad C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(2) - l_3 - \cdots - l_6|.$$

Then  $R$  is a line and  $C^+$  a conic in  $Y^+$ . The blowup  $X^+ = \text{Bl}_{C^+}(Y^+)$  has all desired properties.

**No.15:** Excluded by (3.16.3).

**No.16:** Exists, the construction is as in No.12. Take the complete intersection of two quadrics in  $\mathbb{P}_5$  for  $Y^+$ . Then a general  $S \in |H^+|$  is a del Pezzo surface of degree 4, the blowup of  $\mathbb{P}_2$  in 5 points. Take

$$R \in |\pi^+ \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2| \quad \text{and} \quad C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(2) - l_3 - \cdots - l_5|.$$

Then  $R$  is a line and  $C^+$  a rational curve of degree 3 in  $Y^+$  and define  $X^+ = \text{Bl}_{C^+}(Y^+)$ .

**No.17:** Exists, the construction is as in No.12. Take a smooth Fano threefold of type  $V_{2,5}$  for  $Y^+$  and  $S \in |H^+|$  general. Then  $S$  is the blowup of  $\mathbb{P}_2$  in 4 points. Take

$$R \in |\pi^+ \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2| \quad \text{and} \quad C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(2) - l_3 - l_4|.$$

As above,  $R$  is a line and  $C^+$  a rational curve of degree 4 in  $Y^+$ . Define  $X^+ = \text{Bl}_{C^+}(Y^+)$ .

(3c) Assume  $r = 3$ .

**No.18:** Open.

**No.19:** Excluded by (3.16.3).

**No.20:** Exists, the construction is as No.12. Here  $\tilde{F}^+ = 2L^+ - E^+$  and  $\text{exc}(\psi)$  is a line by (3.16.2). Let  $Y^+$  be a smooth quadric and  $S \subset Y^+$  a general element in  $|2H^+|$ . Then  $S$  is a smooth del Pezzo surface of degree 4, hence the blowup of  $\mathbb{P}_2$  in 5 points. Define

$$R \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2|, \quad \text{and} \quad C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(5) - l_1 - l_2 - 2l_3 - 2l_4 - 2l_5|$$

general. Then  $R + C^+ = -2K_S$ ,  $R \cdot C^+ = 3$ ,  $R$  is a line and  $C^+$  a smooth curve of genus 3 with  $-K_S \cdot C^+ = 7$ . Define  $X^+ = \text{Bl}_{C^+}(Y^+)$ .

To show  $-K_{X^+}$  is nef it is enough to prove  $|\mathcal{I}_{C^+}(3H^+)|$  is base point free and as above it suffices to prove this for  $|\mathcal{I}_{C^+/S} \otimes \mathcal{O}_{Y^+}(3H^+)|_S|$ , which is clear. The pencil on  $X$  is defined by the strict transform of  $|\mathcal{I}_{C^+}(2H^+)|$ , which admits exactly 2 sections and has base locus  $R$ .

**No.21:** Exists, the construction is similar to No.20 above. By (3.16.2) the exceptional locus of  $\psi$  should have degree 2, we take the union of two disjoint lines  $R_1$  and  $R_2$  and define  $C^+ = -2K_S - R_1 - R_2$ . More precisely: take again  $S \in |2H^+|$  general and define

$$R_1 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2|, \quad \text{and} \quad R_2 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_3|.$$

Then  $C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(4) - l_2 - l_3 - 2l_4 - 2l_5|$  general is a smooth elliptic curve of degree 6 intersecting each  $R_i$  in 3 points. The system  $-3K_S - C^+ = -K_S + R_1 + R_2$  is base point free on  $S$  and  $-2K_S - C^+ = R_1 + R_2$  has exactly one section, hence  $|2L^+ - E^+|$  defines the pencil on  $X$  after flopping the strict transforms of  $R_1$  and  $R_2$ .

**No.22:** This case does not exist: by [Mo82],  $X \subset \mathbb{P}(\mathcal{F})$  for some rank 4 vector bundle  $\mathcal{F}$  on  $\mathbb{P}_1$  and  $X \in |2\zeta + \pi^* \mathcal{O}(\mu)|$  for some integer  $\mu$ . By the same argument as in Proposition 3.9, II, 10.), we must have  $\mu = 4$  and  $c_1 = c_1(\mathcal{F}) = -3$ . Then

$$\tilde{E}^+ = \frac{7}{2}(-K_X) - \frac{3}{2}F = 7\zeta|_X + 2F.$$

Then  $h^0(X, \zeta|_X) = h^0(\mathbb{P}_1, \mathcal{F}) = 0$ , contradicting  $c_1(\mathcal{F}) = -3$ .

(3d)  $Y^+$  has index 4, i.e.  $Y^+ = \mathbb{P}_3$ .

**No.23:** Cannot exist since  $F$  is not divisible.

**No.24:** Suppose that  $C^+$  lies on a cubic:

$$H^0(\mathcal{I}_{C^+}(3)) \neq 0.$$

Therefore

$$h^0(3L^+ - E^+) \neq 0 \quad (*)$$

Using  $7(-K_{X^+}) = E^+ + 4\tilde{F}$ , putting in  $L^+$  and dividing by 4 we obtain  $7L^+ = 2E^+ + \tilde{F}$ , hence

$$H^0(7L^+ - 2E^+) = 2.$$

But  $7L^+ - 2E^+ = 2(3L^+ - E^+) + L^+$  which gives via (\*) an inequality  $h^0(7L^+ - 2E^+) \geq 4$ . Hence  $C^+$  does not lie on a cubic. Since  $C^+$  is contained in a quadric (since  $h^0(-K_{X^+}) \neq 0$ ), we may apply a theorem of Gruson-Peskine, see [Ha87], p.151, and obtain  $g(C^+) \leq 15$ , a contradiction.

**No.25:** Open. Here the argument of No.24 does not work.

**No.26:** Here  $3L^+ - E^+ = \tilde{F}$ , hence

$$h^0(\mathcal{I}_{C^+}(3)) = 2.$$

By reasons of degree,  $C^+$  is the intersection of two cubics  $Q_i = \phi_+(\tilde{F}_i)$ . But  $Q_1 \cap Q_2$  must contain the images of curves which are contracted by  $\psi^+$ , hence must contain rational curves. This rules out No.26.

**No.27:** Exists, the construction is the same as No.12. By (3.16.2) the degree of the exceptional curves of  $\psi$  is 1. Take  $S \in |3H^+|$  general. Then  $S$  is a smooth cubic, hence the blowup of  $\mathbb{P}_2$  in 6 points. Take

$$R \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2|, \quad C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(8) - 2l_1 - 2l_2 - 3l_3 - \dots - 3l_6|$$

and define  $X^+ = \text{Bl}_{C^+}(Y^+)$ .



**No.28:** Exists, the construction is as in No.21. By (3.16.2) the degree of the exceptional curves is 2, we take again two disjoint lines  $R_1$  and  $R_2$  in  $S$ , i.e.

$$R_1 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2|, \quad R_2 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_3|.$$

Then  $C^+ \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(7) - 2l_2 - 2l_3 - 3l_4 - 3l_5 - 3l_6|$  is a smooth curve of degree 7 and genus 4 intersecting each  $R_i$  in 4 points. The system  $|-4K_S - C^+|$  is base point free and  $|-3K_S - C^+|$  onedimensional. Define  $X^+ = \text{Bl}_{C^+}(\mathbb{P}_3)$  as usually.

**No.29:** Exists, the construction is as above: by (3.16.2) the exceptional locus of  $\psi$  should have degree 3, hence take 3 disjoint lines  $R_1, R_2$  and  $R_3$  in the cubic  $S \in |3H^+|$ :

$$R_1 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_2|, \quad R_2 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_1 - l_3|, \quad R_3 \in |\pi^* \mathcal{O}_{\mathbb{P}_2}(1) - l_2 - l_3|.$$

Then  $C^+ \in |-3K_S - R_1 - R_2 - R_3|$  general is a smooth elliptic curve of degree 6 intersecting each  $R_i$  in 4 points. The blowup  $X^+ = \text{Bl}_{C^+}(Y^+)$  has all desired properties.  $\square$

#### 4. CONIC BUNDLES

**4.1. Setup.** In this section  $\phi : X \rightarrow Y = \mathbb{P}_2$  denotes a conic bundle with  $\rho(X) = 2$ . As always we assume  $-K_X$  big and nef but not ample and that the anticanonical morphism is small; moreover  $-K_X$  is spanned. The discriminant locus is denoted by  $\Delta$ . Set

$$\tau = \deg \Delta.$$

We introduce the rank 3-bundle

$$\mathcal{E} = \phi_*(-K_X).$$

By [JPR05]  $\mathcal{E}$  is spanned, since  $\psi$  is not divisorial (compare the proof of Proposition 3.2 in [JPR05]). Thus we obtain an embedding

$$X \subset \mathbb{P}(\mathcal{E})$$

such that  $-K_X = \zeta|_X$ . The divisor  $X \subset \mathbb{P}(\mathcal{E})$  is of the form

$$[X] = 2\zeta + \pi^*(\mathcal{O}(\lambda))$$

with some integer  $\lambda$ . Then the adjunction formula yields

$$\lambda = 3 - c_1.$$

Here we use the shorthand  $c_i = c_i(\mathcal{E})$ . Since

$$H^q(\mathbb{P}(\mathcal{E}), -\zeta - \pi^*(\mathcal{O}(\lambda))) = 0$$

for  $q = 0, 1$ , every section in  $H^0(-K_X)$  uniquely lifts to a section of  $\zeta$ . Thus  $|\zeta|$  defines via Stein factorisation a map  $\hat{\psi} : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}'$  extending  $\psi$  and in total a map  $\sigma \circ \hat{\psi} : \mathbb{P}(\mathcal{E}) \rightarrow \hat{W} \subset \mathbb{P}_{g+1}$ .

Now we consider the flopping diagram 2.1.1. *Since the case  $\dim Y^+ = 1$  is already settled by sect. 3, we will always assume that*

$$\dim Y^+ \geq 2,$$

*so that  $Y^+ = \mathbb{P}_2$  (and  $\phi^+$  is a conic bundle) or  $\dim Y^+ = 3$ .*

As usual, we let  $L = \phi^* \mathcal{O}_{\mathbb{P}_1}(1)$  and  $L^+$  be the pull-back to  $X^+$  of the ample generator  $H^+$  on  $Y^+$ . The “strict transform” of  $L$  in  $X^+$  is denoted  $\tilde{L}$  and similarly the strict transform of  $L^+$  in  $X$  is denoted  $\tilde{L}^+$ .

In the case  $\dim Y^+ = 3$  we denote the exceptional divisor of  $\phi^+$  by  $E^+$  and its strict

transform in  $X$  by  $\tilde{E}^+$ . If  $E^+$  contracts to a smooth curve  $C^+$ , we let  $g = g(C^+)$  and  $d = H^+ \cdot C^+$  the genus and degree of  $C^+$ . The index of  $Y^+$  will be  $r$  if  $Y^+$  is Gorenstein; the only non-Gorenstein case occurs when  $E^+ = \mathbb{P}_2$  with normal bundle  $\mathcal{O}(-2)$ . Then  $-2K_{Y^+}$  is Cartier and we define  $r$  by

$$(\phi^+)^*(-2K_{Y^+}) = rL^+.$$

**4.2. Proposition.** *If  $\phi$  is a  $\mathbb{P}_1$ -bundle, then  $r_X = 2$ .*

*Proof.* We write  $X = \mathbb{P}(\mathcal{F})$  with  $\eta$  the tautological line bundle and normalize  $\mathcal{F}$  such that  $c_1(\mathcal{F}) = 0, -1$ . If  $c_1(\mathcal{F}) = -1$ , then  $-K_X = 2\eta + 4L$ , i.e.  $r_X = 2$ . We may hence assume  $c_1(\mathcal{F}) = 0$ .

Consider a curve  $l_\psi$  contracted by  $\psi$  and let  $C = \phi(l_\psi)$ . We may assume that  $C$  is smooth (otherwise normalize). Write

$$\mathcal{F}|_C = \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(-a)$$

and set  $e = 2a$ . Since  $C_0 := l_\psi$  is contractible in  $\mathbb{P}(\mathcal{F}_C)$ , we have  $C_0^2 = -e$ . Since  $-K_X = 2\eta + 3L$  and since  $\eta|_{\mathbb{P}(\mathcal{F}_C)} = C_0 + af$  where  $f = l_\varphi$  is a ruling line, we obtain

$$0 = -K_X \cdot l_\psi = (2C_0 + (2a + 3d)f) \cdot C_0 = -e + 3d,$$

where  $d = \deg(C)$  is the degree of  $C$ .

On the other hand,  $\phi_*(-K_X) = S^2(\mathcal{F}(1))$  not globally generated implies  $\psi$  divisorial by [JPR05], Proposition 3.2. (compare the proof of 3.2., in particular p. 588. There we do not assume  $\psi$  divisorial but show it.). Hence  $\mathcal{F}(1)$  is nef, which gives  $a \leq 1$ . Then  $a = 1$  and  $e = 2 = 3d$ , which is impossible.  $\square$

**4.3. Proposition.** *Assume  $\Delta \neq \emptyset$  and write*

$$\tilde{L} = \alpha^+(-K_{X^+}) + \beta^+L^+ \tag{*}$$

and

$$\tilde{L}^+ = \alpha(-K_X) + \beta L. \tag{**}$$

with  $\alpha^+, \beta^+, \alpha, \beta \in \mathbb{Q}$ .

Then  $\text{Pic}(X) = \mathbb{Z}(-K_X) + \mathbb{Z}L$ , hence  $\alpha, \beta \in \mathbb{Z}$ , and one of the following cases occur.

$$(1) \quad \alpha = 2\alpha^+ \text{ and } \beta^+ = \frac{-1}{2}, \beta = -2;$$

$$(2) \quad \alpha^+ = \alpha \text{ and } \beta^+ = \beta = -1.$$

*Proof.* First note that intersecting with an irreducible component of a reducible conic gives  $\alpha, \beta \in \mathbb{Z}$ , and intersecting with an extremal rational curve of  $\phi^+$  gives  $2\alpha^+, 2\beta^+ \in \mathbb{Z}$ . Moreover,  $\alpha^+, \alpha \geq 0$ . Putting now equation (\*) into (\*\*) and having in mind  $-\tilde{K}_X = -K_{\tilde{X}}$  yields

$$\alpha + \beta\alpha^+ = 0$$

and

$$\beta^+\beta = 1.$$

By symmetry we also have

$$\alpha^+ + \beta^+\alpha = 0.$$

Now a trivial calculation gives (1) and (2).  $\square$

Of course (\*) can be rewritten as

$$L = \alpha^+(-K_X) + \beta^+ \tilde{L}^+$$

and analogously for (\*\*).

We shall also consider a general fiber  $l_\phi$  and also a general fiber  $l_{\phi^+}$  of  $\phi^+$  if  $\dim Y^+ = 2$ . If  $\phi^+$  is birational, we let  $l_{\phi^+}$  be a minimal rational curve contracted by  $\phi^+$ . The intersection numbers with  $-K_X$  resp.  $-K_{X^+}$  are either 1 or 2; for  $l_\phi$  the number is 2. The general  $l_\phi$  will not meet the exceptional locus of  $\psi$ ; thus it lies naturally in  $X^+$ , and we denote the completed family in  $X^+$  by  $l_\phi^+$ . The same for  $l_{\phi^+}$  if  $\dim Y^+ = 2$ ; here the notation is  $\tilde{l}_{\phi^+}$ .

We start with the case that  $\phi^+ : X^+ \rightarrow Y^+ = \mathbb{P}_2$  is a conic bundle.

**4.4. Theorem.** *If  $\phi$  is a proper conic bundle, then the case  $\dim Y^+ = 2$  is impossible.*

*Proof.* Intersect

$$\alpha^+(-K_X) = L + \tilde{L}^+ \quad (*)$$

with a conic  $l_\phi$  to obtain

$$\tilde{L}^+ \cdot l_\phi = 2\alpha^+.$$

Hence

$$L^2 \cdot \tilde{L}^+ = 2\alpha^+.$$

Analogously

$$\tilde{L} \cdot (L^+)^2 = 2\alpha^+.$$

Cube equation (\*), so that

$$(\tilde{L}^+)^3 = (\alpha^+)^3(-K_X)^3 - 12\alpha^+. \quad (**)$$

Now observe that

$$(\tilde{L}^+)^3 < 0.$$

This can be seen either by a spectral sequence argument plus Riemann-Roch, computing  $\chi(\tilde{L}^+)$ , or as follows. Take two general elements

$$S_i \in |\tilde{L}^+|.$$

Then

$$S_1 \cdot S_2 = \sum a_i l_i$$

where  $a_i > 0$  and the  $l_i$  are contracted by  $\psi$ . Since  $\tilde{L}^+ \cdot l_i < 0$ , we obtain  $(\tilde{L}^+)^3 < 0$ . Thus (\*\*) yields

$$(\alpha^+)^2(-K_X)^3 < 12, \quad (***)$$

hence either  $\alpha^+ = 1$  and  $(-K_X)^3 \leq 11$  or  $\alpha^+ = 2$  and  $(-K_X)^3 = 2$ .

Assume  $\alpha^+ = \alpha = 1$ . Then  $-K_X = L + \tilde{L}^+$  and hence

$$-K_X \cdot L^2 = (-K_X)^2 \cdot L - (-K_X) \cdot L \cdot \tilde{L}^+ = (-K_X)^2 \cdot L - L^2 \cdot \tilde{L}^+ - L \cdot (\tilde{L}^+)^2.$$

We obtain  $2 = (12 - \tau) - 4$ , hence  $\tau = 6$ . Analogously we prove  $\tau^+ = 6$ , with  $\tau^+$  the degree of the discriminant locus of  $\phi^+$ . Then  $K_X^2 \cdot (\tilde{L}^+)^2 = (-K_X)^3 - K_X^2 \cdot L$  yields  $(-K_X)^3 = 12$ , contradicting (\*\*).

If  $\alpha^+ = \alpha = 2$ , the analogous computation gives  $\tau = \tau^+ = 9$  and  $(-K_X)^3 = 3$ , which is again impossible.  $\square$

**From now on we assume  $\tau \neq 0$  and  $\phi^+ : X^+ \rightarrow Y^+$  is birational.**

**4.5. Lemma.**  $\phi^+$  cannot be the blow-up of a smooth point.

*Proof.* Assume that  $\phi^+$  is the blow-up of the smooth point  $p$  and let  $E^+$  be the exceptional divisor. Clearly  $E^+$  cannot contain any curve  $l_{\psi^+}$  so that the general line  $l^+ := l_{\phi^+} \subset E^+$  does not meet the exceptional set of  $\psi^+$ . Let  $l' = \psi^+(l^+)$ . Let  $\mathcal{X}' \rightarrow \Delta$  be a smoothing of  $X'$ . Then

$$N_{l'/X'} = \mathcal{O}(-1) \oplus \mathcal{O}(1)$$

and

$$N_{l'/\mathcal{X}'} = \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}.$$

Hence  $l'$  moves to the smooth fibers  $X'_t$ . Let  $l'_t \subset X'_t$  be such a deformation. Then

$$-K_{X'_t} \cdot l'_t = 2$$

so that  $l'_t$  is a conic in the smooth Fano threefold  $X'_t$ . Thus the deformations of  $l'_t$  inside  $X'_t$  fill up  $X'_t$  ( $t \neq 0$ ). But then the deformations of  $l'$  in  $X'$  must fill up  $X'$ , which is absurd.  $\square$

**4.6. Proposition.** The case  $\beta = -2$  is impossible. Moreover  $\alpha^+ = \alpha \in \mathbb{N}$ .

*Proof.* Suppose  $\beta = -2$  so that  $\beta^+ = -\frac{1}{2}$ . First we claim that there cannot be a curve  $C$  contracted by  $\phi^+$  such that  $-K_{X^+} \cdot C = 1$ . In fact, if such a curve exists, then

$$\tilde{L} \cdot C = \alpha^+ - \frac{1}{2}L^+ \cdot C = \alpha^+,$$

hence  $\alpha^+$  is an integer and therefore  $L^+$  is divisible by 2 which is absurd. Thus  $\phi^+$  cannot be a proper conic bundle. It cannot be a  $\mathbb{P}_1$ -bundle either by assumption. So - recalling that we assume  $\dim Y^+ \neq 1$ , the morphism  $\phi^+$  is birational and by Mori's classification the non-existence of a curve  $C$  with  $-K_{X^+} \cdot C = 1$  forces  $\phi^+$  to be the blow-up of a smooth point in  $\tilde{Y}$  which is excluded by the last lemma. Thus  $\beta = -1$  and therefore also  $\beta^+ = -1$ . Consider the decomposition  $\alpha^+(-K_X) = L + \tilde{L}^+$  and intersect with the irreducible component  $l$  of a reducible conic:  $\alpha^+ = L \cdot l + \tilde{L}^+ \cdot l \in \mathbb{N}$ .  $\square$

**4.7. Corollary.** Suppose  $\phi^+$  birational. Then  $\beta^+ = \beta = -1$  and  $\alpha^+ = \alpha \in \mathbb{N}$ . Moreover

$$\tilde{E}^+ = (r\alpha^+ - 1)(-K_X) - rL$$

unless  $E^+ = \mathbb{P}_2$  with normal bundle  $\mathcal{O}(-2)$ . In that case

$$\tilde{E}^+ = (r\alpha^+ - 2)(-K_X) - rL.$$

**4.8. Lemma.** Suppose  $\phi^+$  birational. If  $E^+ \neq \mathbb{P}_2$ , then

$$\begin{aligned} & \left( \frac{(r\alpha^+ - 1)^3}{6} + \frac{(r\alpha^+ - 1)^2}{4} + \frac{(r\alpha^+ - 1)}{12} \right) (-K_X)^3 + (\tau - 12) \left( \frac{(r\alpha^+ - 1)^2 r}{2} + \frac{(r\alpha^+ - 1)r}{2} \right) + \\ & + (r\alpha^+ - 1)(r^2 + 2) + \frac{r^2}{2} - \frac{3}{2}r + 1 \leq 1. \end{aligned}$$

If  $E^+ = \mathbb{P}_2$ , then

$$\left( \frac{(r\alpha^+ - 2)^3}{6} + \frac{(r\alpha^+ - 2)^2}{4} + \frac{(r\alpha^+ - 2)}{12} \right) (-K_X)^3 + (\tau - 12) \left( \frac{(r\alpha^+ - 2)^2 r}{2} + \frac{(r\alpha^+ - 2)r}{2} \right) +$$

$$+(r\alpha^+ - 2)(r^2 + 2) + \frac{r^2}{2} - \frac{3}{2}r + 1 \leq 1.$$

*Proof.* The left hand side of the inequality is just  $\chi(\mathcal{O}_X(\tilde{E}^+))$  (Riemann-Roch and the last corollary). Thus it remains to show that  $\chi(\mathcal{O}_X(\tilde{E}^+)) \leq 1$ . This follows from

$$H^q(\mathcal{O}_X(\tilde{E}^+)) = 0$$

for  $q \geq 2$  which is an easy application of the Leray spectral sequence and the obvious vanishing

$$H^q(\mathcal{O}_{X^+}(E^+)) = 0$$

for  $q \geq 2$ . □

**4.9. Lemma.** *Suppose  $\phi^+$  birational.*

(1) *If  $E^+ = Q_2$ , then*

$$(r(12 - \tau) - 2)(r\alpha^+ - 1) = 2r^2 + 2.$$

(2) *If  $E^+ = \mathbb{P}_2$ , then*

$$(r(12 - \tau) - 1)(r\alpha^+ - 2) = 2r^2 + 2.$$

(3) *If  $E^+$  is ruled, then*

$$(r\alpha^+ - 1)(rd + 2 - 2g - r(12 - \tau)) = (2g - 2) - 2r^2.$$

*Proof.* Let us say that we are in case (1) or (3). Then we compute  $E^3$  in two ways; putting both equations together gives our claim. The first equation is

$$((r\alpha^+ - 1)(-K_X) - E)^3 = 0,$$

the second

$$E^3 = ((r\alpha^+ - 1)(-K_X) - rL)^3.$$

□

**4.10. Lemma.** *Suppose  $\phi^+$  birational.*

(1) *If  $E^+ = Q_2$ , then*

$$(r\alpha^+ - 1)^3(-K_X)^3 - 6(r\alpha^+ - 1)^2 - 6(r\alpha^+ - 1) - 2 < 0.$$

(2) *If  $E^+ = \mathbb{P}_2$ , then*

$$(r\alpha^+ - 2)^3(-K_X)^3 - 3(r\alpha^+ - 2)^2 - 6(r\alpha^+ - 2) - 4 < 0.$$

(3) *If  $E^+$  is ruled, then*

$$(r\alpha^+ - 1)^3(-K_X)^3 - 3(r\alpha^+ - 1)^2(rd + 2 - 2g) + 3(r\alpha^+ - 1)(2g - 2) + rd + 2g - 2 \leq 0.$$

*Proof.* Notice that  $\tilde{L}^3 < 0$ . Then we equate

$$r\tilde{L}^3 = ((r\alpha^+ - 1)(-K_{X^+}) - E^+)^3.$$

□

**4.11. Proposition.** *Suppose  $E^+$  contracts to a point.*

(1) *If  $E^+ = Q_2$ , then  $(-K_{X^+})^3 = 8$ ,  $r = 1$ ,  $(L^+)^3 = 10$ ,  $\tau = 6$  and  $\alpha^+ = 2$ .*

(2) *If  $E^+ = \mathbb{P}_2$ , then  $(-K_{X^+})^3 = 6$ ,  $r = 1$ ,  $(L^+)^3 = 52$ ,  $\tau = 7$  and  $\alpha^+ = 3$ .*

*Both cases really exist.*

*Proof.* First note that  $Y^+$  is singular at the image  $p$  of  $E^+$ . Since  $r = 4$  implies  $Y^+ \simeq \mathbb{P}_3$ , we have  $r \leq 3$ . If  $r = 3$ , then  $Y^+$  must be the quadric cone with vertex  $p$ . But a divisorial resolution of the quadric cone does not have  $\rho = 2$ . So this case is again impossible. We end up with  $r \leq 2$ .

1.) Assume  $E^+ = Q_2$  is a quadric. Set  $a := r\alpha^+ - 1$  for short. Then Lemma 4.10 gives  $a(a^2(-K_{X^+})^3 - 6a - 6) \leq 1$ . But  $a^2(-K_{X^+})^3 - 6a - 6 > 0$  leads to  $(-K_{X^+})^3 = 13$  which is impossible. Hence  $a^2(-K_{X^+})^3 - 6a - 6 \leq 0$ . We find  $a \leq 2$  using  $(-K_{X^+})^3 \geq 4$  since  $X'$  is not hyperelliptic by Corollary 2.10.

We have  $(-K_{X^+})^3 = r^3(L^+)^3 - 2$  and  $Y^+$  has a terminal Gorenstein singularity at  $p$ . This means  $Y^+$  is smoothable, and the pair  $(r, (L^+)^3)$  must correspond to a smooth Fano threefold of Picard number one. Using now the numerical conditions above a short computer program gives the case as stated in the proposition is the only solution.

2.) Assume  $E^+ = \mathbb{P}_2$ . Then  $Y^+$  has a 2-Gorenstein terminal singularity at  $p$  and  $8(-K_{X^+})^3 = r^3(L^+)^3 - 4$ . Setting  $a = r\alpha^+ - 2$  Lemma 4.10 together with  $(-K_{X^+})^3 \geq 4$  gives  $a = 1$  and hence  $r = 1$ . Then  $\tau = 7$  by Lemma 4.9 and finally  $a(-K_X)^3 = (E + rL) \cdot K_X^2 = 1 + r(12 - \tau)$  implies  $(-K_{X^+})^3 = 6$ .

It remains to show the existence.

(1) (i) We start constructing  $X' \subset \mathbb{P}_6$ . Let  $x_0, \dots, x_6$  be homogeneous coordinates of  $\mathbb{P}_6$ ,  $l_0, \dots, l_5$  general linear forms and  $Q$  a general quadric. Then the complete intersection of

$$Q_0 = x_0l_0 + x_1l_1 + x_2l_2, \quad Q_1 = x_0l_3 + x_1l_4 + x_2l_5 \quad \text{and} \quad Q$$

is a Fano threefold  $X'$  of index  $r = 1$  and degree  $(-K_{X'})^3 = 8$  containing the quadric surface  $E' \subset \mathbb{P}_3$  defined by  $Q$  and  $x_0 = x_1 = x_2 = 0$ . A computer calculation (Macaulay) shows  $X'_{\text{sing}} \subset E'$  consists of 6 points (you can also check this directly). Note that the quadrics  $Q_0$  and  $Q_1$  have exactly one singular point not contained in  $Q$ .

(ii) Now construct the first small resolution  $X$ . Blowing up the  $\mathbb{P}_3$  containing  $E'$  in  $\mathbb{P}_6$  resolves the rational map to  $\mathbb{P}_2$  defined by  $x_0, x_1, x_2$ . We obtain

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O}^{\oplus 4} \oplus \mathcal{O}(1)) & \xrightarrow{\psi} & \mathbb{P}_6 \\ \downarrow \phi & & \\ \mathbb{P}_2 & & \end{array}$$

with exceptional divisor  $D$  and tautological line bundle  $\zeta$ . Denote  $F = \phi^*\mathcal{O}_{\mathbb{P}_2}(1)$ . Then  $D \in |\zeta - F|$ . The strict transforms  $\hat{Q}_0, \hat{Q}_1 \in |\zeta + F|$  and  $\hat{Q} = \psi^*Q \in |2\zeta|$  cut out the smooth almost Fano threefold  $X$  with  $-K_X = \zeta|_X$ .

The complete intersection  $\hat{Q}_0 \cap \hat{Q}_1$  is a  $\mathbb{P}_2$ -bundle  $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_2$  defined by an exact sequence

$$0 \longrightarrow \mathcal{O}^{\oplus 2}(-1) \longrightarrow \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(1) \longrightarrow \mathcal{E} \longrightarrow 0$$

on  $\mathbb{P}_2$ . This is a small resolution of the complete intersection  $Q_0 \cap Q_1$ , which is a Fano fourfold of index 3. The singular locus of  $Q_0 \cap Q_1$  is the rational normal curve  $C$  in  $\mathbb{P}_3$ ; the exceptional divisor of  $\mathbb{P}(\mathcal{E}) \rightarrow Q_0 \cap Q_1$  is a  $\mathbb{P}_1$ -bundle over  $C$ . This can be seen as follows:  $\hat{Q}_0$  restricted to  $D = \mathbb{P}(N_{\mathbb{P}_3/\mathbb{P}_6}^*)$  is a section in

$|\zeta + \psi^* \mathcal{O}_{\mathbb{P}_3}(2)|$ , hence a section of  $\mathcal{O}_{\mathbb{P}_3}(1)^{\oplus 3}$ . This gives a  $\mathbb{P}_1$ -bundle over  $\mathbb{P}_3$  outside the only singular point of  $Q_0$ ; here the fiber of  $D \cap \hat{Q}_0$  is a  $\mathbb{P}_2$ .

The intersection with  $\hat{Q}_1$ , a further section in  $\mathcal{O}_{\mathbb{P}_3}(1)^{\oplus 3}$  gives generically an isomorphism with exceptional locus exactly the points, where the map

$$\mathcal{O}_{\mathbb{P}_3}^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}_3}(1)^{\oplus 3}$$

has not rank two. This defines the rational normal curve.

The induced map  $\phi : X \rightarrow \mathbb{P}_2$  is then a proper conic bundle defined by  $\hat{Q}|_{\mathbb{P}(\mathcal{E})} \in |2\zeta|$ , where again  $\zeta$  is the tautological line bundle on  $\mathbb{P}(\mathcal{E})$ . The discriminant  $\Delta$  is the vanishing locus of the determinant of the map

$$\mathcal{E}^* \longrightarrow \mathcal{E}$$

induced by the section  $\hat{Q}$  of  $|2\zeta|$ . We find  $\tau = \deg(\Delta) = 6$ . The intersection of  $Q$  with the rational normal curve  $C$  gives 6 points, the singularities of  $X'$ . The exceptional locus of  $\psi$  consists of 6 single  $\mathbb{P}_1$ 's over these 6 points.

(iii) Finally construct the flop  $X^+$ . First define the strict transform  $\tilde{E}^+$  of the quadric surface  $E' \simeq \mathbb{P}_1 \times \mathbb{P}_1$  in  $\mathbb{P}_6$ . Then  $\tilde{E}^+$  is the blowup of  $E'$  in 6 points, hence a del Pezzo surface of degree 2.

We claim  $X^+ \rightarrow Y^+$  is birational, contracting the strict transform  $E^+ \simeq \mathbb{P}_1 \times \mathbb{P}_1$  of  $\tilde{E}^+$  to a singular point. Note  $-K_X|_{\tilde{E}^+} = \psi^* \mathcal{O}_{\mathbb{P}_2}(1)$ ,  $\psi|_{\tilde{E}^+}$  being the blowdown of the six  $(-1)$ -curves to  $E' \simeq \mathbb{P}_1 \times \mathbb{P}_1$ .

For the normal bundle of the  $\psi$ -exceptional curves  $C_1, \dots, C_6$  we find  $N_{C_i/\tilde{E}^+} = \mathcal{O}_{\mathbb{P}_1}(-1)$ , hence  $N_{C_i/X}$  is of type  $(-1, -1)$  and  $X^+$  may be obtained as simple flop

$$\begin{array}{ccc} & Z = \text{Bl}_{C_1, \dots, C_6}(X) & \\ p \swarrow & & \searrow q \\ X & \dashleftarrow \text{-----} \dashrightarrow & X^+ \end{array}$$

Let  $\hat{E}$  be the strict transform of  $\tilde{E}^+$  in  $Z$ . Then  $\hat{E} \simeq \tilde{E}^+$  and

$$K_Z|_{\hat{E}} = K_X|_{\tilde{E}^+} + \sum C_i = \psi^* \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(-1, -1) + \sum C_i.$$

Blowing down the exceptional divisors of  $p$  the other direction, we obtain  $E^+ = q(\hat{E}) \simeq \mathbb{P}_1 \times \mathbb{P}_1$ , where  $q|_{\hat{E}} = \psi|_{\tilde{E}^+}$ . Let  $K_{X^+}|_{E^+} = \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(a, b)$ . Then using  $q$ , we find

$$K_Z|_{\hat{E}} = q^* K_{X^+}|_{E^+} + \sum C_i = q^* \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(a, b) + \sum C_i.$$

This shows  $a = b = -1$ , and hence  $N_{E^+/X^+} = \mathcal{O}(-1)$  by adjunction. The map contracting  $E^+$  to a point is defined by  $|-2K_{X^+} - F^+|$ , where  $F^+$  is the strict transform of  $F$ . We find  $|-2K_{X^+} - F^+|$  is nef and trivial on  $E^+$ .

(2) (i) We start constructing  $X' \subset \mathbb{P}_5$ . Let  $x_0, \dots, x_5$  be homogeneous coordinates of  $\mathbb{P}_5$ ,  $l_0, l_1, l_2$  general linear forms and  $q_0, q_1, q_2$  three general quadrics. Then the complete intersection of

$$Q = x_0 l_0 + x_1 l_1 + x_2 l_2, \quad \text{and} \quad K = x_0 q_0 + x_1 q_1 + x_2 q_2$$

is a Fano threefold  $X'$  of index  $r = 1$  and degree  $(-K_{X'})^3 = 6$  containing the surface  $E' \simeq \mathbb{P}_2$  defined by  $x_0 = x_1 = x_2 = 0$ . A computer calculation (Macaulay) shows  $X'_{\text{sing}} \subset E'$  consists of 7 points (you can also check this directly).

(ii) Now construct the first small resolution  $X$ . Blowing up  $E' \subset \mathbb{P}_5$  resolves the rational map to  $\mathbb{P}_2$  defined  $x_0, x_1, x_2$ . We obtain

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1)) & \xrightarrow{\psi} & \mathbb{P}_5 \\ \downarrow \phi & & \\ \mathbb{P}_2 & & \end{array}$$

with exceptional divisor  $D$  and tautological line bundle  $\zeta$ . Denote  $F = \phi^* \mathcal{O}_{\mathbb{P}_2}(1)$ . Then  $D \in |\zeta - F|$ . The strict transforms  $\hat{Q} \in |\zeta + F|$  and  $\hat{K} \in |2\zeta + F|$  cut out the smooth almost Fano threefold  $X$  with  $-K_X = \zeta|_X$ .

The  $\mathbb{P}_2$ -bundle  $\hat{Q} \rightarrow \mathbb{P}_2$  is defined by the quotient

$$\mathcal{O}_{\mathbb{P}_2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}_2}(1) \longrightarrow \mathcal{E} = T_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2}(1) \longrightarrow 0,$$

with exceptional divisor  $D_Q = \hat{Q} \cap D$  a  $\mathbb{P}_1$ -bundle over  $E'$ . The induced map  $\phi : X \rightarrow \mathbb{P}_2$  is then a proper conic bundle defined by  $\hat{K}|_{\hat{Q}} \in |2\zeta + F|$ , where again  $\zeta$  is the tautological line bundle on  $\hat{Q} = \mathbb{P}(\mathcal{E})$ . The discriminant  $\Delta$  is the vanishing locus of the determinant of the map

$$\mathcal{E}^* \longrightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_2}(1)$$

induced by the section  $\hat{K}$  of  $|2\zeta + F|$ . We find  $\tau = \deg(\Delta) = 7$ .

The restriction of  $\bar{K}$  to  $D_Q$  is a del Pezzo surface  $E^+$  of degree 2, the blowup of  $E' = \mathbb{P}_2$  in the seven singular points of  $X'$ . Hence  $\psi|_X$  is small, contracting seven rational curves  $C_1, \dots, C_7$  to points.

(iii) Finally construct the flop  $X^+$ . We claim  $X^+ \rightarrow Y^+$  is birational, contracting the strict transform  $E^+ \simeq \mathbb{P}_2$  of  $\tilde{E}^+$  to a singular point. Note  $-K_X|_{\tilde{E}^+} = \psi^* \mathcal{O}_{\mathbb{P}_2}(1)$ ,  $\psi|_{\tilde{E}^+}$  being the blowdown of the seven  $(-1)$ -curves..

For the normal bundle of the  $\psi$ -exceptional curves  $C_1, \dots, C_7$  we find  $N_{C_i/\tilde{E}^+} = \mathcal{O}(-1)$ , hence  $N_{C_i/X}$  is of type  $(-1, -1)$  and  $X^+$  may be obtained as simple flop

$$\begin{array}{ccc} & Z = \mathrm{Bl}_{C_1, \dots, C_7}(X) & \\ p \swarrow & & \searrow q \\ X & \dashrightarrow & X^+ \end{array}$$

Let  $\hat{E}$  be the strict transform of  $\tilde{E}^+$  in  $Z$ . Then  $\hat{E} \simeq \tilde{E}^+$  and

$$K_Z|_{\hat{E}} = K_X|_{\tilde{E}^+} + \sum C_i = \psi^* \mathcal{O}_{\mathbb{P}^2}(-1) + \sum C_i.$$

Blowing down the exceptional divisors of  $p$  the other direction, we obtain  $E^+ = q(\hat{E}) \simeq \mathbb{P}_2$ , where  $q|_{\hat{E}} = \psi|_{\hat{E}^+}$ . Let  $K_{X^+|E^+} = \mathcal{O}_{\mathbb{P}_2}(\lambda)$ . Then using  $q$ , we find

$$K_Z|_{\hat{E}} = q^*K_{X^+}|_{E^+} + \sum C_i = q^*\mathcal{O}_{\mathbb{P}_2}(\lambda) + \sum C_i.$$

This shows  $\lambda = -1$ , and hence  $N_{E^+/X^+} = \mathcal{O}(-2)$  by adjunction. The map contracting  $E^+$  to a point is defined by  $|-3K_{X^+} - F^+|$ , where  $F^+$  is the strict transform of  $F$ . We find  $|-3K_{X^+} - F^+|$  is nef and trivial on  $E^+$ .  $\square$

**4.12. Lemma.** *Suppose  $E^+$  contracts to a curve. Then*

$$(r\alpha^+ - 1)(-K_X)^3 = d + 2 - 2g + r(12 - \tau)$$



and  $\alpha^+ \leq 8$ .

*Proof.* The first claim follows from  $(r\alpha^+ - 1)(-K_X) = \tilde{E}^+ + rL$  together with  $d + 2 - 2g = K_{X^+}^2 \cdot E^+ = K_X^2 \cdot \tilde{E}^+$  and  $K_X^2 \cdot L = 12 - \tau$ .

Using  $12 - \tau \leq 11$  and  $rd + 2 - 2g = r^3(L^+)^3 - rd - (-K_{X^+})^3$  in the above equation gives

$$\alpha^+ \leq \frac{r^2(L^+)^3 + 10}{(-K_{X^+})^3} \leq \frac{r^2(L^+)^3 + 10}{4}.$$

To see the last inequality note that  $X'$  cannot be hyperelliptic by Corollary 2.10. Hence  $(-K_{X^+})^3 \geq 4$ . Now the bounds for  $r$  and  $(L^+)^3$  from Iskovskikh's list prove  $\alpha^+ \leq 8$ .  $\square$

Putting things together numerical computer calculations (written in C) show

**4.13. Proposition.**  *$X$  and  $X^+$  have the following invariants.*

No.	$(-K_X)^3$	$\alpha^+$	$r$	$(L^+)^3$	$d$	$g$	$\tau$
1	4	3	1	8	1	0	7
2	4	4	1	18	8	2	6
3	4	5	1	22	10	2	4
4	4	3	2	4	9	5	7
5	4	4	2	5	11	5	5
6	6	3	1	14	3	0	5
7	6	2	2	3	5	2	7
8	6	3	2	4	6	0	4
9	6	2	3	2	11	10	7
10	8	2	3	2	9	5	5
11	10	2	1	14	1	0	5
12	10	2	2	5	8	2	4
13	10	2	3	2	7	0	3
14	12	2	1	18	2	0	4
15	14	2	1	22	3	0	3
16	18	1	4	1	6	2	4
17	22	1	3	2	5	0	3

**4.14. Theorem.** *If  $\phi^+$  is divisorial contracting  $E^+$  to a curve, then  $X$  is one of 1, 3, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17 in Proposition 4.13.*

*Proof.* We will go case by case through the list in Proposition 4.13.

**No.1:** This is classical and can be found in [IP99], so the existence is clear.

**No.2:** Here we compute

$$\chi(\mathcal{O}_X(\tilde{E}^+)) = 1.$$

By Leray spectral sequence arguments we have

$$H^2(\mathcal{O}_X(\tilde{E}^+)) = H^2(\mathcal{O}_{X'}(E')) = H^2(\mathcal{O}_{X^+}(E^+)) = 0.$$

Since  $H^3(\mathcal{O}_X(\tilde{E}^+)) = 0$  anyway, it follows

$$H^1(\mathcal{O}_X(\tilde{E}^+)) = 0.$$

Again the Leray spectral sequence gives

$$R^1\psi_*(\mathcal{O}_X(\tilde{E}^+)) = 0$$

so that  $E^+ \cdot l = 1$  for all curves contracted by  $\psi^+$ . Hence  $\psi^+|_{E^+}$  is biholomorphic and thus  $E' = \psi^+(E^+)$  is a smooth surface in  $X' \subset \mathbb{P}_4$  (recall  $(-K_X)^3 = 4$ ). Moreover we know that  $E' \subset \mathbb{P}_4$  has degree 6. This contradicts e.g. the double point formula.

**No.3:** Open.

**No.4:** This is parallel to Case 7 below: again  $C^+$  is degenerate, now in  $\mathbb{P}_5$  and the contradiction is the same.

**No.5:** Open.

**No.6:** This is classical as no.1.

**No.7:** Here  $Y^+ \subset \mathbb{P}_4$  is a cubic. Castelnuovo's bound [ACGH85], p.116 implies that  $C^+ \subset \mathbb{P}_4$  is degenerate, so that

$$H^0(\mathcal{I}_{C^+}(1)) \neq 0.$$

Consequently

$$H^0(-K_X - E) \neq 0.$$

On the other hand,  $3(-K_X) = E + 2L$ , hence

$$h^0(-3K_X - E) = 6.$$

This is obviously a contradiction, since  $h^0(-2K_X) > 6$ .

**No.8:** Open.

**No.9:** Here  $Y^+ = Q_3$  is a quadric. We have  $-5K_{X^+} = E^+ + 3\tilde{L}$  and  $-K_{X^+} = 3L^+ - E^+$ . This gives  $\tilde{L} = -K_{X^+} + (2L^+ - E^+)$  on  $X^+$ . Then  $h^0(-K_{X^+}) = 6$  but  $h^0(X^+, \tilde{L}) = 3$  implies

$$H^0(X^+, 2L^+ - E^+) = 0.$$

Now the ideal sequence of  $C^+$  in  $Y^+ = Q_3$  gives

$$0 \longrightarrow H^0(Q_3, \mathcal{O}(2)) \longrightarrow H^0(C^+, \mathcal{O}(2)|_{C^+})$$

is injective, hence  $h^0(C^+, \mathcal{O}(2)) \geq 14$ . On the other hand,  $h^0(C^+, \mathcal{O}_{Q_3}(2)|_C) = 13$  by Riemann–Roch.

**No.10:** Open.

**No.11:** Classical as no.1.

**No.12:** Open.

**No.13:** Open.

**No.14:** Classical as no.1.

**No.15:** Classical as no.1.

**No.16:** We will give a construction. Let  $Y^+ = \mathbb{P}_3$  and  $S \in |\mathcal{O}_{\mathbb{P}_3}(3)|$  a smooth cubic. Write  $\pi: S \rightarrow \mathbb{P}_2$  the blowup of 6 general points in  $\mathbb{P}_2$  and denote the exceptional

curves of  $\pi$  by  $l_1, \dots, l_6$ . Choose

$$C \in |\pi^*\mathcal{O}(4) - 2l_1 - l_2 - \dots - l_5|$$

general. Then  $C \subset \mathbb{P}_3$  is a smooth curve of degree 6 and genus 2. Define  $X^+ = \text{Bl}_C(\mathbb{P}_3)$  with exceptional divisor  $E^+$ . We want to see  $X^+$  is almost Fano.

Let  $L^+$  be the pullback of  $\mathcal{O}_{\mathbb{P}_3}(1)$  and denote the strict transform of  $S$  in  $|3L^+ - E^+|$  on  $X^+$  again by  $S$ . Then  $-K_{X^+} = 4L^+ - E^+$  and we obtain  $(-K_{X^+})^3 = 18$ . Since the restriction map  $H^0(X^+, -K_{X^+}) \rightarrow H^0(S, -K_{X^+}|_S)$  is surjective, it suffices to show that  $-K_{X^+}|_S$  is base point free. We have

$$-K_{X^+}|_S = \mathcal{O}_{\mathbb{P}_3}(4) \otimes \mathcal{O}_S(-C) = \pi^*\mathcal{O}(8) - 2l_1 - 3l_2 - \dots - 3l_5 - 4l_6.$$

Intersecting with the 27 lines on the cubic surface  $S$  shows the claim.

It remains to show that the anticanonical map  $\psi^+$  of  $X^+$  is small and that the flop  $X$  is a conic bundle with discriminant locus of degree 4. Consider

$$\tilde{L} := 3L^+ - E^+.$$

Then  $|\tilde{L}|$  is base point free outside  $S \in |\tilde{L}|$ , and the restriction map

$$0 \rightarrow H^0(X^+, \mathcal{O}_{X^+}) \rightarrow H^0(X^+, \tilde{L}) \rightarrow H^0(S, \tilde{L}|_S) \rightarrow 0$$

is surjective with onedimensional kernel. On  $S$ , we have

$$\tilde{L}|_S = \pi^*\mathcal{O}(5) - l_1 - 2l_2 - \dots - 2l_5 - 3l_6.$$

Subtracting the unique section  $l_\psi^+ \in |\pi^*\mathcal{O}(2) - \sum_{i=2}^6 l_i|$  we get  $l_\phi^+ = \tilde{L}|_S - l_\psi^+$  is a pencil. This shows the base locus of  $\tilde{L}$  is the single rational curve  $l_\psi^+$  and  $h^0(X^+, \tilde{L}) = 3$ .

Flopping  $l_\psi^+$  we therefore obtain a morphism onto  $\mathbb{P}_2$ , which then must be a conic bundle. The degree of the discriminant locus may now be computed easily.

**No.17:** We will give a construction analogously to the last case. Let  $Y^+ = Q_3$  be a smooth quadric and  $S \in |\mathcal{O}_{Q_3}(2)|$  be a general member. Then  $S$  is a smooth del Pezzo surface of degree 4, hence  $\pi: S \rightarrow \mathbb{P}_2$  is the blowup of 5 general points. Denote the exceptional curves of  $\pi$  by  $l_1, \dots, l_5$  as above. Choose

$$C \in |\pi^*\mathcal{O}(2) - l_1|$$

general. Then  $C \subset Q_3$  is a smooth rational curve of degree 5. Define  $X^+ = \text{Bl}_C(Q_3)$  with exceptional divisor  $E^+$ . Denote the strict transform of  $S$  again by  $S$  and the pullback of  $\mathcal{O}_{Q_3}(1)$  by  $L^+$ . Then  $(-K_{X^+})^3 = 22$  and

$$-K_{X^+}|_S = \mathcal{O}_{Q_3}(3)|_S \otimes \mathcal{O}_S(-C) = \pi^*\mathcal{O}(7) - 2l_1 - 3l_2 - \dots - 3l_5.$$

This implies  $-K_{X^+}$  nef since the points are general.

Finally consider  $\tilde{L} := 2L^+ - E^+$ . Then

$$\tilde{L}|_S = \pi^*\mathcal{O}(4) - l_1 - 2l_2 - \dots - 2l_5$$

and subtracting the unique section  $l_\psi^+ \in |\pi^*\mathcal{O}(2) - \sum l_i|$  we get  $l_\phi^+ = \tilde{L}|_S - l_\psi^+$  is a pencil. This shows the base locus of  $\tilde{L}$  is  $l_\psi^+$  and  $h^0(X^+, \tilde{L}) = 3$ . This completes the construction as in the last case.  $\square$

## APPENDIX A. TABLES

Some general notation: let  $-K_X = r_X H$  and  $-K_{X^+} = r_{X^+} H^+$ . For the vector bundle  $\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}_1}(a_i)^{\oplus d_i}$  on  $\mathbb{P}_1$  write  $(a_1^{d_1}, \dots, a_m^{d_m})$  for short. A “+” in the last column indicates that the case exists, a “?” means the existence is unknown.

**A.1. Del Pezzo fibration – Del Pezzo fibration.** Assume  $X$  and  $X^+$  admit del Pezzo fibrations with general fiber  $F$  and  $F^+$ , respectively. Let  $\tilde{F}^+$  be the strict transform of  $F^+$  under the flopping map. Let  $\mathcal{E} = \phi_* H$ ,  $\mathcal{E}^+ = \phi_*^+ H^+$  and assume

$$\tilde{F}^+ = \alpha H + \beta F.$$

We find in any case that  $\mathcal{E}$  and  $\mathcal{E}^+$  are of the same type. Let  $\lambda$  be the maximal integer such that

$$H^0(X, H - \lambda F) \neq 0.$$

No.	$(-K_X)^3$	$r_X$	$K_F^2$	$K_{F^+}^2$	$\mathcal{E}/\mathcal{E}^+$	$\alpha$	$\beta$	$\lambda$	Ref.	$\exists$
1	54	3	9	9	$(0, 1^2)$	1	-1	1	2.12	+
2	32	2	8	8	$(0^2, 1^2)$	1	-1	1	2.13, (1.iii)	+
3	16	2	8	8	$(0^4)$	2	-1	0	2.13, (1.i)	+
4	16	1	8	4	$(0, 1^2, 2^2)$	$\frac{1}{2}$	$-\frac{1}{2}$	2	3.8	+
5	12	1	6	6	$(0^5, 1^2)$	1	-1	1	3.9, I	?
6	10	1	5	5	$(0^4, 1^2)$	1	-1	1	3.9, I	?
7	8	1	4	4	$(0^3, 1^2)$	1	-1	1	3.9, I	+
8	6	1	6	6	$(-1, 0^6)$	2	-1	0	3.9, II	?
9	6	1	3	3	$(0^2, 1^2)$	1	-1	1	3.9, I	+
10	4	1	6	6	$(-1^2, 0^5)$	3	-1	0	3.9, II	?
11	4	1	4	4	$(0^5)$	2	-1	0	3.9, II	+
12	4	1	2	2	$(0, 1^2)$	1	-1	1	3.9, I	+
13	2	1	6	6	$(-1^3, 0^4)$	6	-1	0	3.9, II	?
14	2	1	5	5	$(-1^2, 0^4)$	5	-1	0	3.9, II	?
15	2	1	4	4	$(-1, 0^4)$	4	-1	0	3.9, II	+
16	2	1	3	3	$(0^4)$	3	-1	0	3.9, II	+
17	2	1	1	1	$(1^2)$	1	-1	1	2.5	+

**A.2. Del Pezzo fibration – Conic bundle.** Assume  $X \rightarrow \mathbb{P}_1$  admits a del Pezzo fibration with general fiber  $F$  and  $X^+ \rightarrow \mathbb{P}_2$  is a conic bundle with discriminant locus of degree  $\tau$ . Let  $L^+$  be the pullback of  $\mathcal{O}_{\mathbb{P}_2}(1)$  and  $\tilde{L}^+$  the strict transforms under the flopping map. Then we find in any case

$$\tilde{L}^+ = H - F.$$

Let  $\mathcal{E} = \phi_* H$  and  $\mathcal{F}^+ = \phi_*^+ H^+$  with Chern classes  $c_i = c_i(\mathcal{F}^+)$ .

No.	$(-K_X)^3$	$r_X$	$K_F^2$	$\mathcal{E}$	$\tau$	$(c_1, c_2)$	Ref.	$\exists$
1	40	2	8	$(0, 1^3)$	0	$(3, 4)$	2.13, (1.iv)	+
2	14	1	6	$(0^4, 1^3)$	4	$(5, 13)$	3.11	?
3	12	1	5	$(0^3, 1^3)$	5	$(4, 8)$	3.11	?
4	10	1	4	$(0^2, 1^3)$	6	$(3, 4)$	3.11	+
5	8	1	3	$(0, 1^3)$	7	$(2, 1)$	3.11	+

**A.3. Del Pezzo fibration – Birational contraction with  $\dim(\phi^+(E^+)) = 0$ .**  
Assume  $X \rightarrow \mathbb{P}_1$  admits a del Pezzo fibration with general fiber  $F$  and  $X^+ \rightarrow Y^+$  is a birational contraction with exceptional divisor  $E^+$  contracted to a point. Then  $Y^+$  is a (possibly singular) Fano threefold with  $\rho(Y^+) = 1$ . Let  $L^+$  be the pullback of the generator of  $\text{Pic}(Y^+)$  and  $\tilde{L}^+$  the strict transforms under the flopping map. Then we find in any case

$$\tilde{L}^+ = H - F.$$

Let  $\mathcal{E} = \phi_* H$ .

No.	$(-K_X)^3$	$r_X$	$K_F^2$	$\mathcal{E}$	$(-K_{Y^+})^3$	$(E^+, E^+ _{E^+})$	Ref.	$\exists$
1	24	2	8	$(0^3, 1)$	32	$(\mathbb{P}_2, \mathcal{O}(-1))$	2.13, (1.ii)	+
2	10	1	6	$(0^6, 1)$	18	$(\mathbb{P}_2, \mathcal{O}(-1))$	3.13	+
3	6	1	4	$(0^4, 1)$	8	$(Q, \mathcal{O}(-1))$	3.14	+
4	4	1	3	$(0^3, 1)$	$\frac{9}{2}$	$(\mathbb{P}_2, \mathcal{O}(-2))$	3.15	+

**A.4. Del Pezzo fibration – Birational contraction with  $\dim(\phi^+(E^+)) = 1$ .**  
Assume  $X \rightarrow \mathbb{P}_1$  admits a del Pezzo fibration with general fiber  $F$  and  $X^+ \rightarrow Y^+$  is a birational contraction with exceptional divisor  $E^+$  contracted to a smooth curve of degree  $d$  and genus  $g$ . Then  $r_X = 1$  and  $Y^+$  is a smooth Fano threefold of index  $r_{Y^+}$  with  $\rho(Y^+) = 1$ . Let  $\tilde{E}^+$  be the strict transform of  $E^+$  under the flopping map. Assume

$$\tilde{E}^+ = \alpha H + \beta F.$$

The number in the column “Ref.” refers to the table in Proposition 3.16.

No.	$(-K_X)^3$	$K_F^2$	$r_{Y^+}$	$(-K_{Y^+})^3$	$g$	$d$	$\alpha$	$\beta$	Ref.	$\exists$
1	22	6	2	40	0	4	1	-2	3.16 (17)	+
2	18	5	2	32	0	3	1	-2	3.16 (16)	+
3	18	6	3	54	1	6	2	-3	3.16 (21)	+
4	16	6	4	64	1	6	3	-4	3.16 (29)	+
5	16	5	3	54	3	7	2	-3	3.16 (20)	+
6	14	5	4	64	4	7	3	-4	3.16 (28)	+
7	14	4	2	24	0	2	1	-2	3.16 (14)	+
8	12	4	4	64	7	8	3	-4	3.16 (27)	+
9	10	6	1	16	0	2	1	-1	3.16 (4)	+
10	10	3	2	16	0	1	1	-2	3.16 (12)	+
11	8	6	2	32	1	6	3	-2	3.16 (11)	?
12	8	5	1	12	0	1	1	-1	3.16 (3)	+
13	6	6	1	18	1	6	2	-1	3.16 (2)	?
14	6	5	4	64	8	9	7	-4	3.16 (25)	?
15	4	6	1	16	1	6	3	-1	3.16 (1)	?
16	4	5	2	24	1	5	5	-2	3.16 (8)	?
17	4	5	3	54	9	11	8	-3	3.16 (18)	?

**A.5. Conic bundle – Conic bundle.** Assume  $X \rightarrow \mathbb{P}_2$  and  $X^+ \rightarrow \mathbb{P}_2$  both are conic bundles. Let  $\tau$  and  $\tau^+$  be the degree of their discriminant loci. By Proposition 4.4 then  $\tau = \tau^+ = 0$  and  $r_X = 2$ . This case was treated in [JP06]; we obtain two completely symmetric cases: denote  $\mathcal{F} = \phi_* H$  with Chern classes  $c_i = c_i(\mathcal{F})$  and  $\mathcal{F}^+ = \phi_*^+ H^+$  with Chern classes  $c_i^+ = c_i(\mathcal{F}^+)$ . Then  $c_i = c_i^+$ .

No.	$(-K_X)^3$	$r_X$	$\tau$	$(c_1, c_2)$	Ref.	$\exists$
1	24	2	0	(3, 6)	2.13, (2.iii)	+
2	16	2	0	(3, 7)	2.13, (2.iv)	+

**A.6. Conic bundle – Birational contraction with  $\dim(\phi^+(E^+)) = 0$ .** Assume  $X \rightarrow \mathbb{P}_2$  is a conic bundle,  $\tau$  the degree of the discriminant locus. Denote  $\mathcal{F} = \phi_* H$  with Chern classes  $c_i = c_i(\mathcal{F})$  and  $L = \phi^* \mathcal{O}_{\mathbb{P}_2}(1)$ . Assume  $X^+ \rightarrow Y^+$  is a birational contraction with exceptional divisor  $E^+$  contracted to a point. Then  $Y^+$  is a (possibly singular) Fano threefold with  $\rho(Y^+) = 1$ . Let  $L^+$  be the pullback of the generator of  $\text{Pic}(Y^+)$  and  $\tilde{L}^+$  the strict transform under the flopping map. Assume

$$\tilde{L}^+ = \alpha H + \beta L.$$

No.	$(-K_X)^3$	$r_X$	$\tau$	$(c_1, c_2)$	$(-K_{Y^+})^3$	$(E^+, E^+ _{E^+})$	$\alpha$	$\beta$	Ref.	$\exists$
1	32	2	0	(3, 5)	40	$(\mathbb{P}_2, \mathcal{O}(-1))$	2	-1	2.13, (2.ii)	+
2	8	1	6	(1, 0)	10	$(Q, \mathcal{O}(-1))$	2	-1	4.11	+
3	6	1	7	(1, 0)	52	$(\mathbb{P}_2, \mathcal{O}(-2))$	3	-1	4.11	+

**A.7. Conic bundle – Birational contraction with  $\dim(\phi^+(E^+)) = 1$ .** Assume  $X \rightarrow \mathbb{P}_2$  is a conic bundle,  $\tau$  the degree of the discriminant locus. Denote  $L = \phi^* \mathcal{O}_{\mathbb{P}_2}(1)$ . Assume  $X^+ \rightarrow Y^+$  is a birational contraction with exceptional divisor  $E^+$  contracted to a smooth curve of degree  $d$  and genus  $g$ . Then  $r_X = 1$  and  $Y^+$  is a smooth Fano threefold of index  $r_{Y^+}$  with  $\rho(Y^+) = 1$ . Let  $\tilde{L}^+$  be the strict transform of  $L^+$ , the pullback of the generator of  $\text{Pic}(Y^+)$ , under the flopping map. Then

$$\tilde{L}^+ = \alpha H - L.$$

The number in the column “Ref.” refers to the table in Proposition 4.14.

No.	$(-K_X)^3$	$\tau$	$(-K_{Y^+})^3$	$r_{Y^+}$	$d$	$g$	$\alpha$	Ref.	$\exists$
1	22	3	54	3	5	0	1	4.14 (17)	+
2	18	4	64	4	6	2	1	4.14 (16)	+
3	14	3	22	1	3	0	2	4.14 (15)	+
4	12	4	18	1	2	0	2	4.14 (14)	+
5	10	5	14	1	1	0	2	4.14 (11)	+
6	10	4	40	2	8	2	2	4.14 (12)	?
7	10	3	24	2	7	0	2	4.14 (13)	?
8	8	5	54	3	9	5	2	4.14 (10)	?
9	6	5	14	1	3	0	3	4.14 (6)	+
10	6	4	32	2	6	0	3	4.14 (8)	?
11	4	7	8	1	1	0	3	4.14 (1)	+
12	4	5	40	2	11	5	4	4.14 (5)	?
13	4	4	22	1	10	2	5	4.14 (3)	?

## REFERENCES

- [ACGH85] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris: Geometry of algebraic curves. Springer 1985.
- [BS95] M. Beltrametti, A.J. Sommese: The adjunction theory of complex projective varieties. de Gruyter 1995.
- [Be07] E. Bertini: Introduzione alla geometria proiettiva degli iperspazi. E. Spoerri, Pisa, 1907.
- [Ch99] I. Cheltsov: Three-Dimensional Fano Varieties of Integer Index. Math. Notes **66**, 360-365 (1999).
- [CSP05] I. Cheltsov, C. Shramov, V. Przyjalkowski: Hyperelliptic and trigonal Fano threefolds. Izv. Math. **69**, No.2, 365-421 (2005).
- [Fu90] T. Fujita: Classification theories of polarized varieties. London Math. Soc. Lect. Notes Ser. **155** (1990).

- [Ha87] R. Hartshorne: On the classification of algebraic space curves. II. Proc. Symp. Pure Math. **46**, No.1, 145-164 (1987).
- [Isk78] V.A. Iskovskikh: Fano 3-folds I, II. Math. USSR, Izv. **11**, 485-527 (1977); **12**, 469-506 (1978).
- [Isk80] V.A. Iskovskikh: Anticanonical models of three-dimensional algebraic varieties. J. Soviet Math. **13**, 745-814 (1980).
- [Isk89] V.A. Iskovskikh: Double projection from a line onto Fano threefolds of the first kind. (Eng. Transl) Math. USSR Sb. **66**, 265-284 (1990).
- [IP99] V.A. Iskovskikh, Yu.G. Prokhorov: Algebraic Geometry V: Fano varieties. Springer 1999.
- [JP06] P. Jahnke, T. Peternell: Almost del Pezzo manifolds. math.AG/0612516.
- [JPR05] P. Jahnke, T. Peternell, I. Radloff: Threefolds with big and nef anticanonical bundles I. Math. Ann. **333**, No.3, 569-631 (2005).
- [JR06a] P. Jahnke, I. Radloff: Terminal Fano threefolds and their smoothings. math.AG/0610769.
- [JR06b] P. Jahnke, I. Radloff: Gorenstein Fano threefolds with base points in the anticanonical system. Comp. Math. **142**, No.2, 422-432 (2006).
- [Ko89] J. Kollár: Flops. Nagoya Math. J. **113**, 15-36 (1989).
- [Mo82] S. Mori: Threefolds whose canonical bundles are not numerically effective. Ann. Math. **116**, 133-176 (1982).
- [Na97] Y. Namikawa: Smoothing Fano 3-folds. J. Algebr. Geom. **6**, 307-324 (1997).
- [Shi89] Shin, K.-H.: 3-dimensional Fano Varieties with Canonical Singularities. Tokyo J. Math. **12**, 375-385 (1989).
- [Ta89] K. Takeuchi: Some birational maps of Fano 3-folds. Comp. Math. **71**, 265-284 (1989).

P. JAHNKE - MATHEMATISCHES INSTITUT - UNIVERSITÄT BAYREUTH - D-95440 BAYREUTH, GERMANY

*E-mail address:* `priska.jahnke@uni-bayreuth.de`

TH. PETERNELL - MATHEMATISCHES INSTITUT - UNIVERSITÄT BAYREUTH - D-95440 BAYREUTH, GERMANY

*E-mail address:* `thomas.peternell@uni-bayreuth.de`

I. RADLOFF - MATHEMATISCHES INSTITUT - UNIVERSITÄT BAYREUTH - D-95440 BAYREUTH, GERMANY

*E-mail address:* `ivo.radloff@uni-bayreuth.de`